To an effective local Langlands Correspondence

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Abstract. Let F be a non-Archimedean local field. Let $\mathcal{W}_F$ be the Weil group of F and $\mathcal{P}_F$ the wild inertia subgroup of $\mathcal{W}_F$. Let $\widehat{\mathcal{W}}_F$ be the set of equivalence classes of irreducible smooth representations of $\mathcal{W}_F$. Let $\mathcal{A}^0_n(F)$ denote the set of equivalence classes of irreducible cuspidal representations of $\mathrm{GL}_n(F)$ and set $\widehat{\mathrm{GL}}_F = \bigcup_{n\geqslant 1} \mathcal{A}^0_n(F)$. If $\sigma \in \widehat{\mathcal{W}}_F$, let ${}^L\sigma \in \widehat{\mathrm{GL}}_F$ be the cuspidal representation matched with σ by the Langlands Correspondence. If σ is totally wildly ramified, in that its restriction to $\mathcal{P}_F$ is irreducible, we treat ${}^L\sigma$ as known. From that starting point, we construct an explicit bijection $\mathbb{N} : \widehat{\mathcal{W}}_F \to \widehat{\mathrm{GL}}_F$, sending σ to ${}^N\sigma$. We compare this "naïve correspondence" with the Langlands correspondence and so achieve an effective description of the latter, modulo the totally wildly ramified case. A key tool is a novel operation of "internal twisting" of a suitable representation π (of $\mathcal{W}_F$ or $\mathrm{GL}_n(F)$) by tame characters of a tamely ramified field extension of F, canonically associated to π. We show this operation is preserved by the Langlands correspondence.

We consider the local Langlands correspondence for the general linear group $\mathrm{GL}_n(F)$ over a non-Archimedean local field F of residual characteristic p.

1. Let $\mathcal{W}_F$ be the Weil group of F relative to a chosen separable algebraic closure $\bar{F}/F$. For each integer $n \geqslant 1$, let $\mathcal{G}^0_n(F)$ be the set of equivalence classes of smooth, complex representations of $\mathcal{W}_F$ which are irreducible of dimension n. On the other side, let $\mathcal{A}^0_n(F)$ be the set of equivalence classes of smooth complex representations of $\mathrm{GL}_n(F)$ which are irreducible and cuspidal. The Langlands

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correspondence provides a canonical bijection $\mathcal{G}_n^0(F) \to \mathcal{A}_n^0(F)$, which we denote $\mathbb{L} : \sigma \mapsto {}^L\sigma$.

The irreducible representations of $\mathcal{W}_F$ can be difficult to describe individually in any detail. On the other hand, the irreducible cuspidal representations of $\mathrm{GL}_n(F)$ are the subject of an explicit classification theory [17]. One cannot avoid asking how this wealth of detail on the one side gets translated, via the Langlands correspondence, into concrete information on the other. Our work [10], [11], [13] on *essentially tame* representations revealed the connection in that case to be clear and intuitively obvious (up to a complicated, but functionally minor, technical correction). The aim of this paper is to bring a corresponding degree of clarity to the general case: while the objects under consideration are inherently more complicated, the transparency of the essentially tame case persists to a surprising degree.

2. The set $\mathcal{G}_1^0(F)$ (resp. $\mathcal{A}_1^0(F)$) is the group of smooth characters $\mathcal{W}_F \to \mathbb{C}^\times$ (resp. $F^\times \to \mathbb{C}^\times$), and the Langlands correspondence $\sigma \mapsto {}^L\sigma$ is local class field theory. In the general case $n \geqslant 2$, the correspondence is specified [24] in terms of its behaviour relative to L-functions and local constants of pairs of representations, as in [30], [35]. Its existence is established, in these terms, in [33], [21], [25], [26].

Local constants of pairs of representations remain resistant to explicit computation, so the standard characterization of the correspondence is not helpful as a tool for elucidating it. On the other hand, it does imply that the correspondence is compatible, in a straightforward sense [16], with base change and automorphic induction for cyclic base field extensions [1], [27], [29]. When the cyclic base field extension is also tamely ramified, base change and automorphic induction are, to a useful degree, expressible in terms of the classification of cuspidal representations [3], [4], [8], [9]. This paper is a systematic exploitation of that connection.

3. There is a basic case, outside the scope of this paper. An irreducible smooth representation σ of $\mathcal{W}_F$ is called *totally wildly ramified* if its restriction $\sigma|_{\mathcal{P}_F}$ to the wild inertia subgroup $\mathcal{P}_F$ of $\mathcal{W}_F$ is irreducible. In particular, the dimension of such a representation is a power p^r of p, for some integer $r \geqslant 0$. Little is known of the detailed structure of such representations or their behaviour relative to the

Langlands correspondence. The case $r = 1$ has been worked through in detail in [31] when $p = 2$ (but see also [12]), and in [22] when $p = 3$. For general p, see [32] and [34], plus [4] for a final detail. Some first steps towards the general case $r \geqslant 2$ are made in [4], [5], [6]. We do not pursue that matter any further here. We proceed knowing that ${}^L\sigma$ is defined, but otherwise treat it as effectively unknown. This degree of ignorance has little practical effect, for simple reasons discussed in 7.8 below.

From that initial position, we use explicit, elementary, techniques to construct a "naïve correspondence" $\mathbb{N} : \mathcal{G}_n^0(F) \xrightarrow{\approx} \mathcal{A}_n^0(F)$, sending σ to ${}^N\sigma$. In particular, ${}^N\sigma = {}^L\sigma$ when σ is totally wildly ramified. We estimate, closely and uniformly, the difference between the naïve correspondence and the Langlands correspondence. We thereby reveal a family of transparent and intuitively obvious connections. We simultaneously uncover a novel (and useful) arithmetic property of the Langlands correspondence.

4. We give an overview of the ideas leading to our main results. We start on the Galois side, with a description of the irreducible smooth representations of $\mathcal{W}_F$. Since it determines the flavour of all that follows, we give a detailed summary now but refer to §1 below for proofs.

Let $\widehat{\mathcal{P}}_F$ denote the set of equivalence classes of irreducible smooth representations of $\mathcal{P}_F$. The group $\mathcal{W}_F$ acts on $\widehat{\mathcal{P}}_F$ by conjugation. Taking $\alpha \in \widehat{\mathcal{P}}_F$, the $\mathcal{W}_F$-isotropy group of α is of the form $\mathcal{W}_E$, for a finite, tamely ramified field extension E/F. We use the notation $E = Z_F(\alpha)$, and call E the *F-centralizer field of α.* We write $\mathcal{O}_F(\alpha)$ for the $\mathcal{W}_F$-orbit of α.

Let $\widehat{\mathcal{W}}_F = \bigcup_{n\geqslant 1} \mathcal{G}_n^0(F)$. Starting with $\sigma \in \widehat{\mathcal{W}}_F$, the restriction $\sigma|_{\mathcal{P}_F}$ is a direct sum of irreducible representations α of $\mathcal{P}_F$, all lying in the same $\mathcal{W}_F$-orbit. Thus σ determines an orbit $r_F^1(\sigma) = \mathcal{O}_F(\alpha) \in \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$. If $r_F^1(\sigma)$ consists of the trivial representation of $\mathcal{P}_F$, we say that σ is *tamely ramified.* If it consists of characters of $\mathcal{P}_F$, we say that σ is *essentially tame.*

In the opposite direction, take $\alpha \in \widehat{\mathcal{P}}_F$ and set $E = Z_F(\alpha)$. There then exists $\rho \in \widehat{\mathcal{W}}_E$ such that the restriction of ρ to $\mathcal{P}_E = \mathcal{P}_F$ is equivalent to α. This condition determines ρ up to tensoring with a tamely ramified character of $\mathcal{W}_E$. Let $\sigma \in \widehat{\mathcal{W}}_F$ contain α with multiplicity $m \geqslant 1$, that is, let

$$\dim \mathrm{Hom}_{\mathcal{P}_F}(\alpha, \sigma) = m \geqslant 1.$$

There is then a tamely ramified representation $\tau \in \widehat{\mathcal{W}}_E$, of dimension m, such that

$$\sigma \cong \mathrm{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F}\, \rho \otimes \tau. \tag{1}$$

Here, $\mathrm{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F}$ denotes the functor of *smooth* induction, for which we tend to use the briefer notation $\mathrm{Ind}_{E/F}$. This presentation of σ has strong uniqueness properties.

The expression (1) has an alternative version, often more useful in proofs. Let E_m/E be unramified of degree m. There is then an E_m/E-regular, tamely ramified character ξ of $\mathcal{W}_{E_m}$ such that $\tau \cong \mathrm{Ind}_{E_m/E}\, \xi$. Setting $\rho_m = \rho|_{\mathcal{W}_{E_m}}$, we get

$$\sigma \cong \mathrm{Ind}_{E_m/F}\, \rho_m \otimes \xi. \tag{2}$$

The presentation (1) carries with it another structure. Let $X_1(E)$ (resp. $X_0(E)$) denote the group of tamely ramified (resp. unramified) characters of $E^\times$ or, via class field theory, of $\mathcal{W}_E$. If $\sigma \in \widehat{\mathcal{W}}_F$ contains $\alpha \in \widehat{\mathcal{P}}_F$, we write it in the form (1) and, for $\phi \in X_1(E)$, we define

$$\phi \odot_\alpha \sigma = \mathrm{Ind}_{E/F}\, \phi \otimes \rho \otimes \tau. \tag{3}$$

This gives an action of $X_1(E)$ on the set $\mathcal{G}_m^0(F; \mathcal{O}_F(\alpha))$ of $\sigma \in \widehat{\mathcal{W}}_F$ which contain α with multiplicity m. As the notation indicates, the action does depend on α rather than just $\mathcal{O}_F(\alpha)$. Any other choice of α is of the form α^γ, for some F-automorphism γ of E, and we have the relation

$$\phi^\gamma \odot_{\alpha^\gamma} \sigma = \phi \odot_\alpha \sigma.$$

The case $m = 1$ is of particular interest, since then $\mathcal{G}_1^0(F; \mathcal{O}_F(\alpha))$ is a *principal homogeneous space* over $X_1(E)$.

5. Using the standard classification theory [17], analogous features are not hard to find in the irreducible cuspidal representations of groups $\mathrm{GL}_n(F)$ although it takes rather longer to describe them. We need to use the theory of endo-equivalence classes (or "endo-classes") of simple characters and their tame lifting, as in [3] and the summary in [8] §1. We recall some of the main points.

Let θ be a simple character in $G = \mathrm{GL}_n(F)$, attached to a simple stratum $[\mathfrak{a}, \beta]$ in $A = \mathrm{M}_n(F)$. Let $\varTheta = cl(\theta)$ denote the endo-class of θ. This class does not determine the field $F[\beta]$, but it does determine both the degree $[F[\beta]{:}F]$ and the ramification index $e(F[\beta]|F)$ of the field extension $F[\beta]/F$. We accordingly write $\deg \varTheta = [F[\beta]{:}F]$ and $e(\varTheta) = e(F[\beta]|F)$.

We say that θ is *m-simple* if $\mathfrak{a}$ is maximal among hereditary orders stable under conjugation by $F[\beta]^\times$. Let θ be an m-simple character in G, attached to a simple stratum $[\mathfrak{a}, \beta]$. By definition, θ is a character of the group $H^1_\theta = H^1(\beta, \mathfrak{a})$. The G-normalizer $\boldsymbol{J}_\theta$ of θ is an open, compact modulo centre subgroup of G, having a unique maximal compact subgroup $J^0_\theta = J^0(\beta, \mathfrak{a})$. An *extended maximal simple type over* θ is an irreducible representation $\varLambda$ of $\boldsymbol{J}_\theta$, containing θ, and such that $\varLambda|_{J^0_\theta}$ is a maximal simple type in G, in the sense of [17]. Let $\mathcal{T}(\theta)$ denote the set of equivalence classes of extended maximal simple types over θ.

Taking a different standpoint, let π be an irreducible cuspidal representation of G: in our earlier notation, $\pi \in \mathcal{A}^0_n(F)$. The representation π contains an m-simple character θ_π. Any two choices of θ_π are G-conjugate, hence endo-equivalent. In other words, the endo-class $\vartheta(\pi) = cl(\theta_\pi)$ depends only on π. Certainly, $\deg \vartheta(\pi)$ divides n.

Let $\varTheta$ be an endo-class of degree d dividing n, say $n = md$. Let $\mathcal{A}^0_m(F;\varTheta)$ be the set of $\pi \in \mathcal{A}^0_n(F)$ such that $\vartheta(\pi) = \varTheta$. There is a unique G-conjugacy class of m-simple characters θ in G satisfying $cl(\theta) = \varTheta$. The principal results of [17] assert that, for any such θ, we have a bijection

$$\begin{aligned} \mathcal{T}(\theta) &\longrightarrow \mathcal{A}^0_m(F;\varTheta), \\ \varLambda &\longmapsto c\text{-}\mathrm{Ind}_{\boldsymbol{J}_\theta}^G \varLambda. \end{aligned} \tag{4}$$

6. We introduce a new element of structure. We start with an m-simple character θ in $G = \mathrm{GL}_n(F)$, attached to a simple stratum $[\mathfrak{a}, \beta]$. We let T/F be the maximal tamely ramified sub-extension of the field extension $F[\beta]/F$. The character θ then determines T uniquely, up to *unique* F-isomorphism. We refer to T/F as a *tame parameter field for* θ.

We define an action of $X_1(T)$ on the set $\mathcal{T}(\theta)$. Let G_T denote the G-centralizer of $T^\times$. Thus $G_T \cong \mathrm{GL}_{n_T}(T)$, where $n/n_T = [T{:}F]$. Let $\det_T : G_T \to T^\times$ be the determinant map. Let $\phi \in X_1(T)$. There is a unique character $\phi^{\boldsymbol{J}}$ of $\boldsymbol{J}_\theta$ which

is trivial on $J^1_\theta = J^1(\beta, \mathfrak{a})$ and agrees with $\phi \circ \det_T$ on $G_T \cap \boldsymbol{J}_\theta$. For $\Lambda \in \mathcal{T}(\theta)$, we define

$$\phi \odot \Lambda = \phi^{\boldsymbol{J}} \otimes \Lambda.$$

The representation $\phi \odot \Lambda$ is again an extended maximal simple type over θ, and we have defined an action

$$\begin{aligned} X_1(T) \times \mathcal{T}(\theta) &\longrightarrow \mathcal{T}(\theta), \\ (\phi, \Lambda) &\longmapsto \phi \odot \Lambda, \end{aligned} \tag{5}$$

of the abelian group $X_1(T)$ on the set $\mathcal{T}(\theta)$.

One is tempted to use the induction relation (4) to transfer the action (5) to one of $X_1(T)$ on $\mathcal{A}^0_m(F;\Theta)$. However, the resulting action *depends on the choice of θ within its G-conjugacy class.*

We must therefore proceed more circumspectly, via an external definition of the concept of tame parameter field. An endo-class Φ is called *totally wild* if $\deg \Phi = e(\Phi) = p^r$, for some integer $r \geqslant 0$. Taking Θ as before, a *tame parameter field for Θ* is a finite, tamely ramified field extension E/F such that Θ has a totally wild E/F-lift, the extension E/F being minimal for this property. Such a field E exists, and it is uniquely determined up to F-isomorphism.

We choose an m-simple character θ in $\mathrm{GL}_n(F)$, with tame parameter field T/F and $cl(\theta) = \Theta$. The field T is F-isomorphic to E. The pair (θ, T) determines an endo-class Θ_T, which is a totally wild T/F-lift of Θ: this is the endo-class of the restriction of θ to the group $H^1_\theta \cap G_T$. If $f : E \to T$ is an F-isomorphism, the pull-back $f^*\Theta_T$ is a totally wild E/F-lift of Θ. If we fix a totally wild E/F-lift Ψ of Θ, there is a unique choice of $f : E \to T$ such that $f^*\Theta_T = \Psi$. Combining this with (4) and (5), the resulting action of $X_1(E)$ on $\mathcal{A}^0_m(F;\Theta)$ depends only on Ψ. We accordingly denote it

$$\begin{aligned} X_1(E) \times \mathcal{A}^0_m(F;\Theta) &\longrightarrow \mathcal{A}^0_m(F;\Theta), \\ (\phi, \pi) &\longmapsto \phi \odot_\Psi \pi. \end{aligned}$$

The case $m = 1$ is of particular interest, since then $\mathcal{A}^0_1(F;\Theta)$ becomes a *principal homogeneous space over $X_1(E)$.*

The various actions are related as follows. If Ψ, Ψ' are totally wild E/F-lifts of Θ, there exists an automorphism γ of E such that $\Psi' = \Psi^\gamma$. We then have

$$\phi^\gamma \odot_{\Psi^\gamma} \pi = \phi \odot_\Psi \pi,$$

for all $\phi \in X_1(E)$ and all $\pi \in \mathcal{A}^0_m(F;\Theta)$. All of these actions extend the standard action of twisting by characters of $F^\times$: if $\chi \in X_1(F)$ and $\chi_E = \chi \circ \mathrm{N}_{E/F}$, then

$$\chi_E \odot_\Psi \pi = \chi\pi : g \longmapsto \chi(\det g)\,\pi(g).$$

Overall, these discussions attach to $\pi \in \mathcal{A}^0_n(F)$ an endo-class $\Theta = \vartheta(\pi)$, a finite tame extension E/F, an integer $m = n/\deg\Theta$ and, relative to the choice of a totally wild E/F-lift Ψ of Θ, an action $\odot_\Psi$ of $X_1(E)$ on $\mathcal{A}^0_m(F;\Theta)$. In particular, $\mathcal{A}^0_1(F;\Theta)$ is a principal homogeneous space over $X_1(E)$. All of this is, at least superficially, parallel to the structures uncovered on the Galois side.

7. We may now start to build a connection between the two sides. The foundation is a result from [8], generalizing the ramification theorem of local class field theory. Let $\mathcal{E}(F)$ denote the set of endo-classes of simple characters over F.

Ramification Theorem. *There is a unique bijection* $\Phi_F : \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F \to \mathcal{E}(F)$ *such that* $\vartheta({}^L\sigma) = \Phi_F(r^1_F(\sigma))$, *for all* $\sigma \in \widehat{\mathcal{W}}_F$.

This result expresses the relationships between the Langlands Correspondence, tamely ramified automorphic induction and tame lifting for endo-classes. It relies on the explicit formula [15] for the *conductor* of a pair of representations. We show here that the map Φ_F is compatible with tamely ramified base field extension: if K/F is a finite tame extension and $\alpha \in \widehat{\mathcal{P}}_F$, then $\Phi_K(\alpha)$ is a K/F-lift of $\Phi_F(\alpha)$. We deduce:

Tame Parameter Theorem. *Let* $\alpha \in \widehat{\mathcal{P}}_F$ *and write* $\Phi_F(\alpha) = \Phi_F(\mathcal{O}_F(\alpha))$.

(1) *The field* $Z_F(\alpha)/F$ *is a tame parameter field for* $\Phi_F(\alpha)$ *and*

$$\deg\Phi_F(\alpha) = [Z_F(\alpha){:}F]\,\dim\alpha.$$

(2) *For each* $m \geqslant 1$, *the Langlands correspondence induces a bijection*

$$\mathcal{G}^0_m(F;\mathcal{O}_F(\alpha)) \longrightarrow \mathcal{A}^0_m(F;\Phi_F(\alpha)). \tag{6}$$

We therefore concentrate on analyzing the bijection (6) of the theorem.

8. We return to the presentation (2) of an element σ of $\mathcal{G}^0_m(F;\mathcal{O}_F(\alpha))$. In particular, $\alpha \in \widehat{\mathcal{P}}_F$ has F-centralizer field E/F. We take the unramified extension E_m/E, in $\bar F$, of degree m and set $\Delta = \mathrm{Gal}(E_m/E)$. The orbit $\mathcal{O}_{E_m}(\alpha)$ is just $\{\alpha\}$, so we abbreviate $\mathcal{G}^0_1(E_m;\mathcal{O}_{E_m}(\alpha)) = \mathcal{G}^0_1(E_m;\alpha)$. Note that all representations $\nu \in \mathcal{G}^0_1(E_m;\alpha)$ are *totally wildly ramified.*

The group Δ acts on $\mathcal{G}^0_1(E_m;\alpha)$ in a natural way; let $\mathcal{G}^0_1(E_m;\alpha)^{\Delta\text{-reg}}$ denote the subset of Δ-regular elements. As in (2) above, we have a canonical bijection

$$\mathrm{Ind}_{E_m/F} : \Delta\backslash\mathcal{G}^0_1(E_m;\alpha)^{\Delta\text{-reg}} \xrightarrow{\approx} \mathcal{G}^0_m(F;\mathcal{O}_F(\alpha)).$$

If we take $\phi \in X_1(E)$ and write $\phi_m = \phi\circ \mathrm{N}_{E_m/E}$, this bijection satisfies

$$\mathrm{Ind}_{E_m/F}\,\phi_m\otimes\nu = \phi\odot_\alpha \mathrm{Ind}_{E_m/F}\,\nu, \quad \nu\in\mathcal{G}^0_1(E_m;\alpha)^{\Delta\text{-reg}},$$

by definition.

Moving to the other side, set $\Psi = \Phi_E(\alpha)$ and $\Psi_m = \Phi_{E_m}(\alpha)$. Thus Ψ is a totally wild E/F-lift of $\Theta = \Phi_F(\alpha)$, and E/F is a tame parameter field for Θ. The endo-class Ψ_m is the unique E_m/E-lift of Ψ. The natural action of Δ on endo-classes over E_m fixes Ψ_m, so Δ acts on $\mathcal{A}^0_1(E_m;\Psi_m)$. The Langlands correspondence induces a Δ-bijection $\mathcal{G}^0_1(E_m;\alpha)\to\mathcal{A}^0_1(E_m;\Psi_m)$ such that

$${}^L(\phi_m\otimes\nu) = \phi_m\,{}^L\nu.$$

The heart of the paper is the explicit construction in §5 of a canonical bijection

$$\mathit{ind}_{E_m/F} : \Delta\backslash\mathcal{A}^0_1(E_m;\Psi_m)^{\Delta\text{-reg}} \xrightarrow{\approx} \mathcal{A}^0_m(F;\Theta)$$

such that

$$\mathit{ind}_{E_m/F}(\phi_m\rho) = \phi\odot_\Psi \mathit{ind}_{E_m/F}\,\rho,$$

for $\rho\in\mathcal{A}^0_1(E_m;\Psi_m)^{\Delta\text{-reg}}$ and $\phi\in X_1(E)$. This is defined in terms of extended maximal simple types, using a developed version of the Glauberman correspondence [20] from the character theory of finite groups. It is elementary and completely explicit in nature, but we say no more of it in this introductory essay.

Immediately, there is a unique bijection

$$\begin{aligned}\mathbb{N} : \mathcal{G}^0_m(F;\mathcal{O}_F(\alpha)) &\longrightarrow \mathcal{A}^0_m(F;\Theta),\\ \sigma &\longmapsto {}^N\sigma,\end{aligned}$$

such that the following diagram commutes.

$$(\star) \qquad \begin{array}{ccc} \mathcal{G}_1^0(E_m;\alpha)^{\Delta\text{-reg}} & \xrightarrow{\mathbb{L}} & \mathcal{A}_1^0(E_m;\Psi_m)^{\Delta\text{-reg}} \\ {\scriptstyle \mathrm{Ind}_{E_m/F}}\big\downarrow & & \big\downarrow{\scriptstyle ind_{E_m/F}} \\ \mathcal{G}_m^0(F;\mathcal{O}_F(\alpha)) & \xrightarrow[\mathbb{N}]{} & \mathcal{A}_m^0(F;\Theta). \end{array}$$

This *naïve correspondence* $\mathbb{N}$ further satisfies

$$^N(\phi \odot_\alpha \sigma) = \phi \odot_\Psi {}^N\sigma,$$

for $\phi \in X_1(E)$, $\sigma \in \mathcal{G}_m^0(F;\mathcal{O}_F(\alpha))$, and where $\Psi = \Phi_E(\alpha)$.

9. Our main result compares the two bijections $\mathcal{G}_m^0(F;\mathcal{O}_F(\alpha)) \to \mathcal{A}_m^0(F;\Theta)$ induced by, respectively, the Langlands correspondence $\sigma \mapsto {}^L\sigma$ and the naïve correspondence $\sigma \mapsto {}^N\sigma$.

Let $X_0(E)_m$ denote the group of $\chi \in X_0(E)$ for which $\chi^m = 1$.

Comparison Theorem. *Let $\alpha \in \widehat{\mathcal{P}}_F$ have F-centralizer field E/F and let $m \geqslant 1$ be an integer. There exists a character $\mu = \mu_{m,\alpha}^F \in X_1(E)$ such that*

$$^L\sigma = \mu \odot_{\Phi_E(\alpha)} {}^N\sigma,$$

for all $\sigma \in \mathcal{G}_m^0(F;\mathcal{O}_F(\alpha))$. The character $\mu_{m,\alpha}^F$ is uniquely determined modulo $X_0(E)_m$.

We note that, if $\chi \in X_0(E)_m$ and $\pi \in \mathcal{A}_m^0(F;\Phi_F(\alpha))$, then $\chi \odot_{\Phi_E(\alpha)} \pi = \pi$. The Comparison Theorem has the following immediate corollary, tying together the two "interior twisting" operations.

Homogeneity Theorem. *If $\sigma \in \mathcal{G}_m^0(F;\mathcal{O}_F(\alpha))$ and $\phi \in X_1(E)$, then*

$$^L(\phi \odot_\alpha \sigma) = \phi \odot_{\Phi_E(\alpha)} {}^L\sigma.$$

The third of our main results is the Types Theorem of 7.6. If $\sigma \in \widehat{\mathcal{W}}_F$ and $\pi = {}^L\sigma$, then π contains a maximal simple type. The Types Theorem allows one to read off the structure of this type from the presentation (1) of σ.

10. In the essentially tame case, we gave a complete account of the "discrepancy character" $\mu^F_{m,\alpha}$, purely in terms of the simple stratum underlying the endo-class $\Phi_F(\alpha)$ [10], [11] [13]. Even there, it poses challenging problems and we make no such attempt here. In the general case, we determine its restriction to units of E: this is the essence of the Types Theorem. Consideration of central characters yields its restriction to $(F^\times)^{m\dim\alpha}$ (7.3.2). The character $\mu^F_{m,\alpha}$ of $E^\times$ is thereby determined up to an unramified factor of degree dividing $me(E|F)\dim\alpha$ (and $\mu^F_{m,\alpha}$ is only defined modulo unramified characters of order dividing m). In principle, therefore, it is amenable to description in terms of local constants of a finite number of pairs, using [7].

11. We indicate briefly the strategy of the proof. This follows the essentially tame case quite closely, but the technical hurdles are somewhat higher.

The diagram $(\star)$ has an analogue relative to the Langlands correspondence: there is a unique bijection

$$\mathrm{a}_{E_m/F} : \varDelta\backslash\mathcal{A}^0_1(E_m;\varPsi_m)^{\varDelta\text{-reg}} \xrightarrow{\approx} \mathcal{A}^0_m(F;\varTheta)$$

such that the diagram

$$(\dagger)\qquad \begin{array}{ccc} \mathcal{G}^0_1(E_m;\alpha)^{\varDelta\text{-reg}} & \xrightarrow{\mathbb{L}} & \mathcal{A}^0_1(E_m;\varPsi_m)^{\varDelta\text{-reg}} \\ {\scriptstyle \mathrm{Ind}_{E_m/F}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{a}_{E_m/F}} \\ \mathcal{G}^0_m(F;\mathcal{O}_F(\alpha)) & \xrightarrow[\mathbb{L}]{} & \mathcal{A}^0_m(F;\varTheta). \end{array}$$

commutes. We have to compare $\mathrm{a}_{E_m/F}$ with $ind_{E_m/F}$.

The field extension E_m/F admits a tower of subfields

$$E_m = L_0 \supset L_1 \supset \cdots \supset L_r \supset L_{r+1} = F,$$

in which

(1) L_r/F is unramified;
(2) L_i/L_{i+1} is cyclic and totally ramified of prime degree, $1 \leqslant i \leqslant r-1$;
(3) the group of L_1-automorphisms of L_0 is trivial.

The map $\mathrm{a}_{E_m/F}$ has a corresponding factorization

$$\mathrm{a}_{E_m/F} = \mathrm{A}_{L_r/F} \circ \mathrm{A}_{L_{r-1}/L_r} \circ \ldots \mathrm{A}_{L_1/L_2} \circ \mathrm{a}_{L_0/L_1},$$

in which each map labelled A is *automorphic induction,* in the sense of [27], [28], [29]. The first factor, a_{L_0/L_1}, is defined by the diagram (†) relative to the base field L_1 in place of F.

Comparison of a_{L_0/L_1} with ind_{L_0/L_1} is straightforward. One sees easily that these two maps differ by the $\odot$-twist with a character $\nu_{L_0/L_1} \circ \mathrm{N}_{E_m/E}$, independent of the representations under consideration. However, the method gives no information about this character. From then on, we work inductively, comparing $ind_{L_0/L_{i+1}}$ with $\mathrm{A}_{L_i/L_{i+1}} \circ ind_{L_0/L_i}$. We have to show that these two maps differ by the $\odot$-twist with a character $\nu_{L_i/L_{i+1}} \in X_1(E)$. The discrepancy character μ is then the product of these characters ν.

We give an exact, rather than informative, expression for each of the characters $\nu_{L_i/L_{i+1}}$, $1 \leqslant i \leqslant r$. When the sub-extension in question is totally ramified, that is, when $1 \leqslant i \leqslant r{-}1$, the formula (8.9 Corollary) involves an induction constant and transfer factor arising from the automorphic induction equation, along with some symplectic signs coming from the Glauberman correspondence. While we make no attempt to evaluate it in general, a simple exercise shows, for instance, that the character is trivial when $p[L_i{:}L_{i+1}]$ is odd. For the unramified extension L_{r-1}/L_r, the corresponding character has order $\leqslant 2$. We give an explicit formula (10.7 Corollary) involving only elementary quantities and some symplectic signs.

Our handling of the automorphic induction equation, for a tame cyclic extension K/F, directly generalizes the method of [10], [13]. It requires the Uniform Induction Theorem of [13], [28] to control an induction constant. Since we can only rely on rather weaker linear independence properties of characters, we have to compensate by a closer investigation of transfer factors. These do not seem amenable to direct computation, so we have had to evolve some novel methods. Likewise, we have had to develop a more structural approach to the symplectic signs.

Note on characteristic. We shall often refer to our earlier papers [3], [4] and their successors, in which we imposed the hypothesis that F be of characteristic zero. The only reason for doing that was the lack of a theory of base change and automorphic induction in positive characteristic. Such material is now available in [29], so the results of all of our earlier papers in the area apply equally in

positive characteristic.

Notation and conventions. The following will be standard throughout, and used without further elaboration. If F is a non-Archimedean local field, we denote by $\mathfrak{o}_F$ the discrete valuation ring in F, by $\mathfrak{p}_F$ the maximal ideal of $\mathfrak{o}_F$, and by $\Bbbk_F$ the residue class field $\mathfrak{o}_F/\mathfrak{p}_F$. The characteristic of $\Bbbk_F$ is always denoted p. We set $U_F = U_F^0 = \mathfrak{o}_F^\times$ and $U_F^k = 1+\mathfrak{p}_F^k$, $k \geqslant 1$. We write $\boldsymbol{\mu}_F$ for the group of roots of unity in F of order prime to p. In particular, $U_F = \boldsymbol{\mu}_F \times U_F^1$ and reduction modulo $\mathfrak{p}_F$ induces an isomorphism $\boldsymbol{\mu}_F \cong \Bbbk_F^\times$.

If E/F is a finite field extension, we denote by $\mathrm{N}_{E/F} : E^\times \to F^\times$ and $\mathrm{Tr}_{E/F} : E \to F$ the norm and trace maps respectively. We write $\mathrm{Aut}(E|F)$ for the group of F-automorphisms of the field E. If E/F is Galois, then $\mathrm{Aut}(E|F) = \mathrm{Gal}(E/F)$.

We fix, once for all, a separable algebraic closure $\bar F/F$ and let $\mathcal{W}_F$ be the Weil group of $\bar F/F$. We let $\mathcal{I}_F$, $\mathcal{P}_F$ denote respectively the inertia subgroup and the wild inertia subgroup of $\mathcal{W}_F$. If E/F is a finite separable extension with $E \subset \bar F$, we identify the Weil group $\mathcal{W}_E$ of $\bar F/E$ with the group of elements of $\mathcal{W}_F$ which fix E.

We denote by $X_0(F)$ (resp. $X_1(F)$) the group of smooth characters $\mathcal{W}_F \to \mathbb{C}^\times$ which are unramified (resp. tamely ramified) in the sense of being trivial on $\mathcal{I}_F$ (resp. $\mathcal{P}_F$).

We let $\boldsymbol{a}_F : \mathcal{W}_F \to F^\times$ be the Artin Reciprocity map, normalized to take *geometric* Frobenius elements of $\mathcal{W}_F$ to prime elements of F. We use $\boldsymbol{a}_F$ to identify the group of smooth characters of $\mathcal{W}_F$ with the group of smooth characters of $F^\times$. In particular, we identify $X_0(F)$ (resp. $X_1(F)$) with the group of characters of $F^\times$ which are trivial on U_F (resp. U_F^1). If $m \geqslant 1$ is an integer, then $X_0(F)_m$ is the group of $\chi \in X_0(F)$ such that $\chi^m = 1$.

Let $n \geqslant 1$ be an integer; let $A = \mathrm{M}_n(F)$ (the algebra of $n\times n$ matrices over F) and $G = \mathrm{GL}_n(F)$. If $\mathfrak{a}$ is a hereditary $\mathfrak{o}_F$-order in A, with Jacobson radical $\mathrm{rad}\,\mathfrak{a} = \mathfrak{p}_\mathfrak{a}$, we write $U_\mathfrak{a} = U_\mathfrak{a}^0 = \mathfrak{a}^\times$, and $U_\mathfrak{a}^k = 1+\mathfrak{p}_\mathfrak{a}^k$, $k \geqslant 1$. We write $\mathcal{K}_\mathfrak{a} = \{x \in G : x^{-1}\mathfrak{a}x = \mathfrak{a}\}$. The group $\mathcal{K}_\mathfrak{a}$ is also the G-normalizer of $U_\mathfrak{a}$.

If Δ is a group acting on a set X, we denote by X^Δ the set of Δ-fixed points in X. An element x of X is *Δ-regular* if its Δ-isotropy is trivial. We denote by

$X^{\Delta\text{-reg}}$ the set of Δ-regular elements of X.

1. Representations of Weil groups

We give a systematic account of the irreducible smooth representations of the locally profinite group $\mathcal{W}_F$, in terms of restriction to the wild inertia subgroup $\mathcal{P}_F$ of $\mathcal{W}_F$. The material of the section, as far as the end of 1.4, is basically well-known but we need to have all of the relevant details in one place.

Let E/F be a finite field extension, with $E \subset \bar{F}$. If σ, resp. ρ, is an irreducible smooth representation of $\mathcal{W}_F$, resp. $\mathcal{W}_E$, then both the smoothly induced representation $\operatorname{Ind}_{E/F} \rho = \operatorname{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F} \rho$, and the restricted representation $\sigma|_{\mathcal{W}_E}$, are finite-dimensional and *semisimple, cf.* [12] 28.7 Lemma. Consequently, we may work entirely within the category of finite-dimensional, smooth, semisimple representations of $\mathcal{W}_F$, using the elementary methods of Clifford-Mackey theory.

1.1. Let $n \geqslant 1$ be an integer. Let $\mathcal{G}_n^0(F)$ denote the set of equivalence classes of irreducible smooth representations of $\mathcal{W}_F$ of dimension n. We set

$$\widehat{\mathcal{W}}_F = \bigcup_{n\geqslant 1} \mathcal{G}_n^0(F).$$

If σ is a smooth representation of $\mathcal{W}_F$, we say that σ is *unramified* (resp. *tamely ramified*) if $\operatorname{Ker}\sigma$ contains $\mathcal{I}_F$ (resp. $\mathcal{P}_F$).

1.2. Let $\widehat{\mathcal{P}}_F$ denote the set of equivalence classes of irreducible smooth representations of the profinite group $\mathcal{P}_F$. The group $\mathcal{W}_F$ acts on $\widehat{\mathcal{P}}_F$ by conjugation.

Let $\alpha \in \widehat{\mathcal{P}}_F$. Since $\mathcal{P}_F$ is a pro-p group, the group $\alpha(\mathcal{P}_F)$ is finite. The dimension of α is finite and of the form p^r, for an integer $r \geqslant 0$. Let

$$N_F(\alpha) = \{g \in \mathcal{W}_F : \alpha^g \cong \alpha\},$$

where α^g denotes the representation $x \mapsto \alpha(gxg^{-1})$ of $\mathcal{P}_F$.

Proposition. *Let $\alpha \in \widehat{\mathcal{P}}_F$. There is a finite, tamely ramified field extension E/F, with $E \subset \bar{F}$, such that $N_F(\alpha) = \mathcal{W}_E$.*

Proof. Since $\dim\alpha$ is finite, the smooth representation of $\mathcal{W}_F$ *compactly* induced by α is finitely generated over $\mathcal{W}_F$. It therefore admits an irreducible quotient,

σ say. The dimension of σ is finite (see 28.6 Lemma 1 of [12]), so $\sigma|_{\mathcal{P}_F}$ is a finite direct sum of irreducible representations, all conjugate to α. Moreover, any $\mathcal{W}_F$-conjugate of α occurs here. Thus $N_F(\alpha)$ is closed and of finite index in $\mathcal{W}_F$. It follows that $N_F(\alpha)$ is open in $\mathcal{W}_F$, of finite index and containing $\mathcal{P}_F$. That is, $N_F(\alpha) = \mathcal{W}_E$, for a finite, tamely ramified extension E/F inside $\bar{F}$. □

In the situation of the proposition, we call E the F*-centralizer field* of α, and denote it $Z_F(\alpha)$.

1.3. We consider extension properties of a given representation $\alpha \in \widehat{\mathcal{P}}_F$.

Let $E = Z_F(\alpha)$. If K/F is a finite, tamely ramified extension, inside $\bar{F}$, such that α admits extension to a representation of $\mathcal{W}_K$, then surely $K \supset E$. In the opposite direction, we prove:

Proposition. *Let $\alpha \in \widehat{\mathcal{P}}_F$, and let $E = Z_F(\alpha)$.*

(1) *There exists an irreducible smooth representation ρ of $\mathcal{W}_E$ such that*
   (a) $\rho|_{\mathcal{P}_F} \cong \alpha$, *and*
   (b) $\det \rho|_{\mathcal{I}_F}$ *has p-power order.*
(2) *If ρ' is a smooth representation of $\mathcal{W}_E$ such that $\rho'|_{\mathcal{P}_F} \cong \alpha$, then there exists a unique character $\chi \in X_1(E)$ such that $\rho' \cong \rho \otimes \chi$.*

Proof. We first concentrate on the uniqueness properties.

Lemma 1.

(1) *Let θ be an irreducible smooth representation of $\mathcal{I}_F$ such that $\theta|_{\mathcal{P}_F}$ is irreducible. If ψ is a character of $\mathcal{I}_F$, trivial on $\mathcal{P}_F$, such that $\theta \otimes \psi \cong \theta$, then $\psi = 1$.*
(2) *Let τ be an irreducible smooth representation of $\mathcal{W}_F$ such that $\tau|_{\mathcal{P}_F}$ is irreducible. If ψ is a tamely ramified character of $\mathcal{W}_F$, such that $\tau \otimes \psi \cong \tau$, then $\psi = 1$.*

Proof. In part (1), we view θ and $\psi \otimes \theta$ as acting on the same vector space V, say. By hypothesis, there is a map $f \in \mathrm{Aut}_{\mathbb{C}}(V)$ such that $f \circ \theta(g) = \psi(g)\theta(g) \circ f$, for all $g \in \mathcal{I}_F$. In particular, $f \circ \theta(x) = \theta(x) \circ f$, for $x \in \mathcal{P}_F$. Since $\theta|_{\mathcal{P}_F}$ is irreducible, the map f is a non-zero scalar, whence ψ is trivial, as required. The proof of (2) is identical. □

Lemma 2.

(1) *There exists a unique smooth representation ρ_α of $\mathcal{J}_E$ such that*

  (a) $\rho_\alpha|_{\mathcal{P}_F} \cong \alpha$, *and*

  (b) *the character* $\det \rho_\alpha$ *has finite p-power order.*

(2) *If ρ' is an irreducible smooth representation of $\mathcal{J}_E$ such that $\rho'|_{\mathcal{P}_F}$ contains α, then $\rho' \cong \rho_\alpha \otimes \psi$, for a unique character ψ of $\mathcal{J}_E/\mathcal{P}_F$.*

Proof. Let τ be an irreducible smooth representation of $\mathcal{J}_E$ such that $\tau|_{\mathcal{P}_F}$ contains α. Thus $\tau|_{\mathcal{P}_F}$ is a multiple of α. If $\mathcal{K} = \operatorname{Ker}\tau$, then α surely extends uniquely to a representation τ_1 of $\mathcal{P}_F\mathcal{K}$ which is trivial on $\mathcal{K}$. The restriction $\tau|_{\mathcal{P}_F\mathcal{K}}$ is thus a multiple of τ_1. The quotient $\mathcal{J}_E/\mathcal{P}_F\mathcal{K}$ is finite cyclic, and τ_1 is stable under conjugation by $\mathcal{J}_E$. Therefore τ_1 admits extension to a representation, $\tilde\tau_1$ say, of $\mathcal{J}_E$. The irreducible representations of $\mathcal{J}_E$ containing τ_1 are then of the form $\chi \otimes \tilde\tau_1$, where χ ranges over the characters of $\mathcal{J}_E$ trivial on $\mathcal{P}_F\mathcal{K}$. In particular, $\tau|_{\mathcal{P}_F\mathcal{K}} \cong \tau_1$ and so $\tau|_{\mathcal{P}_F} \cong \alpha$.

This argument also shows that, if τ' is an irreducible smooth representation of $\mathcal{J}_E$ containing α, then $\tau' \cong \tau \otimes \psi$, for a character ψ of $\mathcal{J}_E/\mathcal{P}_F$. Consider the determinant character

$$\det(\tau \otimes \psi) = \psi^{\dim \tau} \det \tau. \tag{1.3.1}$$

The dimension $\dim\tau = \dim\alpha$ is a power of p, the character $\det\tau$ has finite order, while ψ has finite order not divisible by p. Thus there is a unique choice of ψ such that $\det(\tau \otimes \psi)$ has p-power order. This proves (1).

In (2) the existence assertion has already been proved. The uniqueness property is given by Lemma 1. □

We prove the proposition. The uniqueness property of ρ_α (as in Lemma 2) implies that ρ_α is stable under conjugation by $\mathcal{W}_E$. The group $\mathcal{W}_E/\mathcal{J}_E$ is infinite cyclic and discrete, so ρ_α admits extension to a smooth representation ρ of $\mathcal{W}_E$, as required for part (1). For the same reason, any irreducible representation of $\mathcal{W}_E$ containing ρ_α is of the form $\psi \otimes \rho$, for a character ψ of $\mathcal{W}_E/\mathcal{P}_F$, as required for part (2). The uniqueness assertion of (2) is given by Lemma 1. □

1.4. To analyze the irreducible smooth representations of $\mathcal{W}_F$, we introduce a class of objects generalizing the admissible pairs of [10]. (See below, 1.6 Remark, for the precise connection.)

Definition. *An* admissible datum *over F is a triple $(E/F, \rho, \tau)$ satisfying the following conditions.*

(1) *E/F is a finite, tamely ramified field extension with $E \subset \bar{F}$.*
(2) *ρ is an irreducible smooth representation of $\mathcal{W}_E$ such that the restriction $\alpha = \rho|_{\mathcal{P}_F}$ is irreducible and $Z_F(\alpha) = E$.*
(3) *τ is an irreducible, smooth, tamely ramified representation of $\mathcal{W}_E$.*

Two admissible data $(E_i/F, \rho_i, \tau_i)$ over F, $i = 1, 2$, are deemed equivalent if there exist $g \in \mathcal{W}_F$ and $\chi \in X_1(E_2)$ such that $E_2 = E_1^g$, $\rho_2 \cong \rho_1^g \otimes \chi$ and $\tau_2 \cong \tau_1^g \otimes \chi^{-1}$. Let $\mathfrak{G}(F)$ denote the set of equivalence classes of admissible data over F.

If $(E/F, \rho, \tau)$ is an admissible datum over F, we define

$$\Sigma(\rho, \tau) = \operatorname{Ind}_{E/F} \rho \otimes \tau. \tag{1.4.1}$$

Surely $\Sigma(\rho, \tau)$ depends only on the equivalence class of the datum $(E/F, \rho, \tau)$. The main result of the section is the following.

Theorem. *If $(E/F, \rho, \tau)$ is an admissible datum over F, the representation $\Sigma(\rho, \tau)$ of $\mathcal{W}_F$ is irreducible. The map $\Sigma : \mathfrak{G}(F) \to \widehat{\mathcal{W}}_F$ is a bijection.*

Proof. Let $(E/F, \rho, \tau)$ be an admissible datum. We prove that $\operatorname{Ind}_{E/F} \rho \otimes \tau$ is irreducible. Set $\alpha = \rho|_{\mathcal{P}_F}$. The tamely ramified representation $\tau \in \widehat{\mathcal{W}}_E$ is of the form $\tau = \operatorname{Ind}_{K/E} \chi$, where K/E is a finite, unramified extension and $\chi \in X_1(K)$ is such that the conjugates χ^g, $g \in \mathcal{W}_K \backslash \mathcal{W}_E$, are distinct. Writing $\rho_K = \rho|_{\mathcal{W}_K}$, we have $\rho \otimes \tau \cong \operatorname{Ind}_{K/F} \rho_K \otimes \chi$. Let $x \in \mathcal{W}_F$ intertwine $\rho_K \otimes \chi$. It also intertwines $\alpha = (\rho_K \otimes \chi)|_{\mathcal{P}_F}$, and therefore lies in $\mathcal{W}_E$. The conjugate representation $(\rho_K \otimes \chi)^x = \rho_K \otimes \chi^x$ is equivalent to $\rho_K \otimes \chi$. Part (2) of 1.3 Lemma 1 implies $\chi^x = \chi$, whence $x \in \mathcal{W}_K$. Thus

$$\operatorname{Ind}_{E/F} \rho \otimes \tau \cong \operatorname{Ind}_{K/F} \rho_K \otimes \chi$$

is irreducible, as required for the first assertion of the theorem. This same argument also proves:

Lemma. *Let τ, τ' be irreducible, tamely ramified representations of $\mathcal{W}_E$ of the same dimension. If the representations $\rho \otimes \tau$, $\rho \otimes \tau'$ are intertwined by an element of $\mathcal{W}_F$, then $\tau \cong \tau'$.*

We next show that the map $\Sigma : \mathfrak{G}(F) \to \widehat{\mathcal{W}}_F$ is injective. Let $(E/F, \rho, \tau)$, $(E'/F, \rho', \tau')$ be admissible data with the same image. The representations $\alpha = \rho|_{\mathcal{P}_F}$, $\alpha' = \rho'|_{\mathcal{P}_F}$ are therefore $\mathcal{W}_F$-conjugate. Replacing $(E'/F, \rho', \tau')$ by a conjugate, which does not affect its equivalence class, we may assume $E' = E$ and $\alpha' = \alpha$. By 1.3 Proposition, $\rho' \cong \rho \otimes \psi$, for some tamely ramified character ψ of $\mathcal{W}_E$. Adjusting $(E/F, \rho', \tau')$ in its equivalence class, we can take $\rho' = \rho$. The lemma now implies $\tau' \cong \tau$, as required.

It remains to show that the map Σ is surjective. Let $\sigma \in \widehat{\mathcal{W}}_F$, let α be an irreducible component of $\sigma|_{\mathcal{P}_F}$, and set $E = Z_F(\alpha)$. Let θ denote the natural representation of $\mathcal{W}_E$ on the α-isotypic subspace of σ. Let θ_0 be an irreducible component of θ. Any element of $\mathcal{W}_F$ which intertwines θ_0 also intertwines α and so lies in $\mathcal{W}_E$. This shows that $\mathrm{Ind}_{E/F}\,\theta_0$ is irreducible, hence equivalent to σ. The Mackey induction formula further shows that $\theta_0 = \theta$.

By its definition, the restriction of θ to $\mathcal{P}_F$ is a direct sum of copies of α. By part (2) of 1.3 Lemma 2, the restriction of θ to $\mathcal{I}_E$ is a direct sum of representations $\rho_\alpha \otimes \psi$, with ρ_α as in that lemma and various characters ψ of $\mathcal{I}_E/\mathcal{P}_F$. Moreover, if we choose such an ψ, an element $x \in \mathcal{W}_E$ intertwines $\rho \otimes \psi$ if and only if $\psi^x \cong \psi$. The $\mathcal{W}_E$-stabilizer of ψ is of the form $\mathcal{W}_K$, where K/E is finite and unramified. Since ρ_α admits extension to a representation ρ of $\mathcal{W}_E$ (1.3 Proposition) and $\mathcal{W}_K/\mathcal{I}_E$ is cyclic, the representation $\rho_\alpha \otimes \psi$ extends to a representation $\rho|_{\mathcal{W}_K} \otimes \Psi$ of $\mathcal{W}_K$, occurring in $\theta|_{\mathcal{W}_K}$. We then have $\theta \cong \rho \otimes \tau$, where $\tau = \mathrm{Ind}_{K/E}\,\Psi$. The triple $(E/F, \rho, \tau)$ is an admissible datum and $\Sigma(\rho, \tau) \cong \sigma$, as required.

This completes the proof of the theorem. $\square$

1.5. Let $\mathcal{O} \in \mathcal{W}_F \backslash \widehat{\mathcal{P}}_F$, say $\mathcal{O}$ is the $\mathcal{W}_F$-orbit $\mathcal{O}_F(\alpha)$ of $\alpha \in \widehat{\mathcal{P}}_F$. We set

$$d(\mathcal{O}) = d_F(\alpha) = [Z_F(\alpha){:}F]\,\dim\alpha. \tag{1.5.1}$$

If $m \geqslant 1$ is an integer, we define $\mathcal{G}_m^0(F; \mathcal{O})$ to be the set of $\sigma \in \mathcal{G}_{md(\mathcal{O})}^0(F)$ such that $\sigma|_{\mathcal{P}_F}$ contains α. From 1.4 Theorem, we deduce:

Corollary. *Let m be a positive integer, let $\alpha \in \widehat{\mathcal{P}}_F$ and write $\mathcal{O} = \mathcal{O}_F(\alpha)$. For $\sigma \in \widehat{\mathcal{W}}_F$, the following are equivalent:*

(1) $\sigma \in \mathcal{G}^0_m(F; \mathcal{O})$;
(2) *there is an admissible datum $(E/F, \rho, \tau)$ such that $\rho|_{\mathcal{P}_F} \cong \alpha$, $\dim \tau = m$ and $\sigma \cong \Sigma(\rho, \tau)$;*
(3) $\dim \mathrm{Hom}_{\mathcal{P}_F}(\alpha, \sigma) = m$.

The theorem also reveals a useful family of structures on the set $\widehat{\mathcal{W}}_F$.

Taking $\alpha \in \widehat{\mathcal{P}}_F$ and $\mathcal{O} = \mathcal{O}_F(\alpha)$ as before, set $E = Z_F(\alpha)$. For $\sigma \in \mathcal{G}^0_m(F; \mathcal{O})$ and $\phi \in X_1(E)$, we define a representation $\phi \odot_\alpha \sigma \in \mathcal{G}^0_m(F; \mathcal{O})$ as follows. We write $\sigma = \Sigma(\rho, \tau)$, for an admissible datum $(E/F, \rho, \tau)$ with $\rho|_{\mathcal{P}_F} \cong \alpha$. The triple $(E/F, \phi\otimes\rho, \tau)$ is again an admissible datum. We put

$$\phi \odot_\alpha \sigma = \Sigma(\phi\otimes\rho, \tau) = \Sigma(\rho, \phi\otimes\tau). \tag{1.5.2}$$

This does not depend on the choice of datum $(E/F, \rho, \tau)$ satisfying the stated conditions. The pairing $(\phi, \sigma) \mapsto \phi \odot_\alpha \sigma$ endows $\mathcal{G}^0_m(F; \mathcal{O})$ with the structure of $X_1(E)$-space, in that

$$\phi\phi' \odot_\alpha \sigma = \phi \odot_\alpha (\phi' \odot_\alpha \sigma),$$

for $\phi, \phi' \in X_1(E)$, $\sigma \in \mathcal{G}^0_m(F; \mathcal{O})$.

This structure *does* depend on α rather than on $\mathcal{O}$ and E. For, if $\alpha' \in \mathcal{O}$ and $Z_F(\alpha') = E$, then $\alpha' = \alpha^\gamma$, for some $\gamma \in \mathrm{Aut}(E|F)$. We then get the relation

$$\phi^\gamma \odot_{\alpha^\gamma} \sigma \cong \phi \odot_\alpha \sigma, \tag{1.5.3}$$

for $\phi \in X_1(E)$, $\sigma \in \mathcal{G}^0_m(F; \mathcal{O})$. (Clearly, the relation (1.5.3) holds equally for any $\gamma \in \mathcal{W}_F$.)

Working for the moment with base field E, the $\mathcal{W}_E$-orbit of α is $\{\alpha\}$ and so we write $\mathcal{G}^0_m(E; \alpha)$ rather than $\mathcal{G}^0_m(E; \{\alpha\})$. In this case, the definition (1.5.2) reduces to

$$\phi \odot_\alpha \sigma = \phi \otimes \sigma, \tag{1.5.4}$$

for $\sigma \in \mathcal{G}^0_m(E; \alpha)$ and $\phi \in X_1(E)$.

Proposition. *The map*

$$\begin{aligned}\mathcal{G}^0_m(E;\alpha) &\longrightarrow \mathcal{G}^0_m(F;\mathcal{O})\\ \sigma &\longmapsto \operatorname{Ind}_{E/F}\sigma,\end{aligned}$$

is a bijection satisfying

$$\operatorname{Ind}_{E/F}\phi\otimes\sigma = \phi\odot_\alpha \operatorname{Ind}_{E/F}\sigma,\quad \phi\in X_1(E),\ \sigma\in\mathcal{G}^0_m(E;\alpha). \tag{1.5.5}$$

In particular, $\mathcal{G}^0_1(F;\mathcal{O})$ is a principal homogeneous space over $X_1(E)$.

Proof. The first assertion follows from 1.4 Theorem and the definitions. The set $\mathcal{G}^0_1(E;\alpha)$ is a principal homogeneous space over $X_1(E)$, by 1.3 Proposition. The second assertion now follows from the first. □

1.6. For some purposes, a variant of 1.4 Theorem is technically convenient.

Let $\alpha\in\widehat{\mathcal{P}}_F$ and set $E = Z_F(\alpha)$. Let E_m/E be unramified of degree m and set $\Delta = \mathrm{Gal}(E_m/E)$. The group Δ then acts on the set $\mathcal{G}^0_1(E_m;\alpha)$. Let $\mathcal{G}^0_1(E_m;\alpha)^{\Delta\text{-reg}}$ denote the subset of Δ-regular elements. (As usual, an element of a Δ-set is called Δ-regular if its Δ-isotropy group is trivial.)

If $\phi\in X_1(E)$, write $\phi_m = \phi\circ \mathrm{N}_{E_m/E}$. The group $X_1(E)$ thus acts on $\mathcal{G}^0_1(E_m;\alpha)$ by $(\phi,\zeta)\mapsto\phi_m\otimes\zeta$. The subset $\mathcal{G}^0_1(E_m;\alpha)^{\Delta\text{-reg}}$ is stable under this action.

Proposition. *If $\zeta\in\mathcal{G}^0_1(E_m;\alpha)^{\Delta\text{-reg}}$, the representation $\operatorname{Ind}_{E_m/F}\zeta$ is irreducible and lies in $\mathcal{G}^0_m(F;\mathcal{O})$, $\mathcal{O}=\mathcal{O}_F(\alpha)$. Moreover,*

$$\operatorname{Ind}_{E_m/F}\phi_m\otimes\zeta = \phi\odot_\alpha\operatorname{Ind}_{E_m/F}\zeta,$$

for $\phi\in X_1(E)$ and $\zeta\in\mathcal{G}^0_1(E_m;\alpha)^{\Delta\text{-reg}}$. The induced map

$$\operatorname{Ind}_{E_m/F}:\Delta\backslash\mathcal{G}^0_1(E_m;\alpha)^{\Delta\text{-reg}}\longrightarrow\mathcal{G}^0_m(F;\mathcal{O})$$

is bijective.

Proof. Choose $\rho\in\mathcal{G}^0_1(E;\alpha)$ and set $\rho_m = \rho|\mathcal{W}_{E_m}$.

Lemma. *The map*

$$X_1(E_m) \longrightarrow \mathcal{G}_1^0(E_m;\alpha),$$
$$\xi \longmapsto \xi\otimes\rho_m,$$

is a bijective Δ-map.

Proof. The lemma follows directly from 1.3 Proposition. □

In particular, $\xi\otimes\rho_m$ is Δ-regular if and only if ξ is Δ-regular, and this condition is equivalent to $\mathrm{Ind}_{E_m/E}\,\xi$ being irreducible. We have

$$\mathrm{Ind}_{E_m/E}\,\xi\otimes\rho_m \cong \rho\otimes\mathrm{Ind}_{E_m/E}\,\xi.$$

Thus $\mathrm{Ind}_{E_m/E}$ induces a bijection

$$\Delta\backslash\mathcal{G}_1^0(E_m;\alpha)^{\Delta\text{-reg}} \xrightarrow{\approx} \mathcal{G}_m^0(E;\alpha).$$

The proposition now follows from 1.5 Proposition. □

Remark. Suppose $\dim\alpha = 1$ and let $\zeta\in\mathcal{G}_1^0(E_m;\alpha)$. Thus $\dim\zeta = 1$ and we may view ζ as a character of $E_m^\times$, via local class field theory. The representation ζ lies in $\mathcal{G}_1^0(E_m;\alpha)^{\Delta\text{-reg}}$ if and only if the pair $(E_m/F,\zeta)$ is *admissible,* in the sense of [10].

2. Simple characters and tame parameters

We recall the theory of simple characters [17] and tame lifting [3]. We orient the treatment so as to give a degree of prominence to a certain tamely ramified field extension attached to a simple character θ. The invariance properties of this *tame parameter field* provide the only novelty here. The main result (2.7) is rather technical, but essential for what follows.

We follow the notational schemes of [17] and [3], making some minor modifications to indicate more clearly the interdependences between various objects. The notation introduced here, particularly that in 2.3, will be standard for the rest of the paper.

2.1. Let $n \geqslant 1$ be an integer, and set $A = \mathrm{M}_n(F)$, $G = \mathrm{GL}_n(F)$. We consider a simple stratum in A of the form $[\mathfrak{a}, l, 0, \beta]$ (as in [17] Chapter 1). The positive integer l is determined by the relation $\beta\mathfrak{a} = \mathfrak{p}_\mathfrak{a}^{-l}$, where $\mathfrak{p}_\mathfrak{a}$ is the Jacobson radical of $\mathfrak{a}$, so we tend to omit it from the notation and abbreviate $[\mathfrak{a}, l, 0, \beta] = [\mathfrak{a}, \beta]$.

As in [17] Chapter 3, the simple stratum $[\mathfrak{a}, \beta]$ determines compact open subgroups

$$H^1(\beta, \mathfrak{a}) \subset J^1(\beta, \mathfrak{a}) \subset J^0(\beta, \mathfrak{a})$$

of $U_\mathfrak{a}$, such that $J^1(\beta, \mathfrak{a}) = J^0(\beta, \mathfrak{a}) \cap U_\mathfrak{a}^1$. Let A_β denote the centralizer of β in A. Thus A_β is isomorphic to a full matrix algebra over the field $F[\beta]$ and $\mathfrak{a}_\beta = \mathfrak{a} \cap A_\beta$ is a hereditary $\mathfrak{o}_{F[\beta]}$-order in A_β. We set

$$\boldsymbol{J}(\beta, \mathfrak{a}) = \mathcal{K}_{\mathfrak{a}_\beta} \cdot J^1(\beta, \mathfrak{a}).$$

A smooth character ψ_F of F, of level one, attaches to the stratum $[\mathfrak{a}, \beta]$ a non-empty set $\mathcal{C}(\mathfrak{a}, \beta, \psi_F)$ of distinguished characters of the group $H^1(\beta, \mathfrak{a})$, as in [17] 3.2. These are known as *simple characters.* Since ψ_F will be fixed throughout, we usually omit it from the notation. We assemble a variety of facts about this situation, using [17] 3.3.2, 3.5.1.

(2.1.1) *Let $[\mathfrak{a}, \beta]$ be a simple stratum in A and let $\theta \in \mathcal{C}(\mathfrak{a}, \beta, \psi_F)$. Let $I_G(\theta)$ denote the set of elements of G which intertwine θ and let $\boldsymbol{J}_\theta$ be the group of $g \in G$ which normalize θ.*

(1) *The group $\boldsymbol{J}_\theta$ is open and compact modulo centre in G. Moreover,*
  - (a) *the group $\boldsymbol{J}_\theta$ has a unique maximal compact subgroup J_θ^0,*
  - (b) *the group J_θ^0 has a unique maximal, normal pro p-subgroup J_θ^1, and*
  - (c) *the set $I_G(\theta)$ is $J_\theta^1 G_\beta J_\theta^1 = J_\theta^0 G_\beta J_\theta^0$, where G_β denotes the G-centralizer of β.*

(2) *If $[\mathfrak{a}', \beta']$ is a simple stratum in A such that $\theta \in \mathcal{C}(\mathfrak{a}', \beta', \psi_F)$, then*
  - (a) *$\mathfrak{a}' = \mathfrak{a}$,*
  - (b) *$e(F[\beta']|F) = e(F[\beta]|F)$, $f(F[\beta']|F) = f(F[\beta]|F)$,*
  - (c) *$J_\theta^1 = J^1(\beta', \mathfrak{a}')$, $J_\theta^0 = J^0(\beta', \mathfrak{a}')$, and*
  - (d) *$\boldsymbol{J}_\theta = \boldsymbol{J}(\beta', \mathfrak{a}')$.*

In particular, the conclusions (c), (d) of (2) hold for $[\mathfrak{a}', \beta'] = [\mathfrak{a}, \beta]$.

If $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$, for a simple stratum $[\mathfrak{a}, \beta]$, we say that $F[\beta]/F$ is a *parameter field* for θ. The isomorphism class of $F[\beta]/F$ is *not* necessarily determined by θ, as the following example shows.

Example. Take $G = \mathrm{GL}_p(F)$, $A = \mathrm{M}_p(F)$, and let $\mathfrak{a}$ be a *minimal* hereditary $\mathfrak{o}_F$-order in A. We fix a simple stratum $[\mathfrak{a}, 1, 0, \alpha]$ in A (to use the full notation for once). If E/F is a totally ramified extension of degree p, there is an embedding $\boldsymbol{\iota} : E \to A$ such that $\boldsymbol{\iota} E^\times \subset \mathcal{K}_\mathfrak{a}$. There exists $\beta \in E$ such that $[\mathfrak{a}, 1, 0, \boldsymbol{\iota}\beta]$ is equivalent to $[\mathfrak{a}, 1, 0, \alpha]$, that is, $\boldsymbol{\iota}\beta \equiv \alpha \pmod{\mathfrak{a}}$. In particular, $\mathcal{C}(\mathfrak{a}, \alpha) = \mathcal{C}(\mathfrak{a}, \boldsymbol{\iota}\beta)$. So, if $\theta \in \mathcal{C}(\mathfrak{a}, \alpha)$, then *every* totally ramified extension E/F, of degree p, is a parameter field for θ.

Returning to the general situation, it is necessary to extend the framework to include the notion of a *trivial* simple character: a trivial simple character is the trivial character of $U^1_\mathfrak{a}$, for a hereditary $\mathfrak{o}_F$-order $\mathfrak{a}$ in $\mathrm{M}_n(F)$, for some $n \geqslant 1$. If θ is such a character, we define $\boldsymbol{J}_\theta$, J^0_θ, J^1_θ as in (2.1.1), to obtain $\boldsymbol{J}_\theta = \mathcal{K}_\mathfrak{a}$, $J^0_\theta = U_\mathfrak{a}$, $J^1_\theta = U^1_\mathfrak{a}$. We may treat this case within the same framework, simply by taking the view that $F[\beta] = F$.

2.2. We consider the class of all simple characters $\theta \in \mathcal{C}(\mathfrak{a}, \beta, \psi_F)$, as $[\mathfrak{a}, \beta]$ ranges over all simple strata in all matrix algebras $\mathrm{M}_n(F)$, $n \geqslant 1$. As in [3], this class carries an equivalence relation, called *endo-equivalence,* extending the relation of conjugacy (see (2.2.4) below).

We let $\mathcal{E}(F)$ denote the set consisting of all endo-equivalence classes of simple characters, together with a trivial element $\mathbf{0}_F$ which is the class of trivial simple characters.

If $\Theta \in \mathcal{E}(F)$ and $\theta \in \Theta$, we say that θ is a *realization* of Θ and that Θ is the *endo-class* of θ. We use the notation $\Theta = cl(\theta)$. From the definition of endo-equivalence [3] and (2.1.1), we obtain:

(2.2.1) *Let $\Theta \in \mathcal{E}(F)$, $\Theta \neq \mathbf{0}_F$. If $\theta_i \in \mathcal{C}(\mathfrak{a}_i, \beta_i)$ is a realization of Θ, for $i = 1, 2$, then*

$$e(F[\beta_1]|F) = e(F[\beta_2]|F), \quad f(F[\beta_1]|F) = f(F[\beta_2]|F),$$

and hence $[F[\beta_1]{:}F] = [F[\beta_2]{:}F]$.

In this situation, we set

$$e(\Theta) = e(F[\beta_i]|F), \quad f(\Theta) = f(F[\beta_i]|F), \quad \deg \Theta = [F[\beta_i]{:}F]. \tag{2.2.2}$$

By convention,

$$e(\mathbf{0}_F) = f(\mathbf{0}_F) = \deg \mathbf{0}_F = 1. \tag{2.2.3}$$

If $\Theta \in \mathcal{E}(F)$, $\Theta \neq \mathbf{0}_F$, and if $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ is a realization of Θ, we say that $F[\beta]/F$ is a *parameter field* for Θ. By convention, F is the only parameter field for $\mathbf{0}_F$. In general, however, the parameter field is not uniquely determined (as in the Example of 2.1).

Finally, we recall [17] 3.5.11, [3] 8.7:

(2.2.4) *For $i = 1, 2$, let $[\mathfrak{a}, \beta_i]$ be a simple stratum in $\mathrm{M}_n(F)$ and let $\theta_i \in \mathcal{C}(\mathfrak{a}, \beta_i)$. The following conditions are equivalent:*

(1) *θ_1 intertwines with θ_2 in G;*
(2) *θ_1 is conjugate to θ_2 in G;*
(3) *θ_1 is endo-equivalent to θ_2.*

2.3. We recall from [3] some fundamental properties of endo-classes of simple characters, relative to tamely ramified base field extension. We take the opportunity to fix some notation for the rest of the paper.

From [3] 7.1, 7.7, we obtain the following facts.

(2.3.1) *Let $[\mathfrak{a}, \beta]$ be a simple stratum in $A = \mathrm{M}_n(F)$. Let K/F be a subfield of A satisfying the following conditions:*

(a) *K/F is tamely ramified,*
(b) *K commutes with β, and*
(c) *the K-algebra $K[\beta]$ is a subfield of A such that $K[\beta]^\times \subset \mathcal{K}_\mathfrak{a}$.*

Let A_K be the A-centralizer of K, and set $G_K = A_K^\times$, $\mathfrak{a}_K = \mathfrak{a} \cap A_K$.

(1) *The pair $[\mathfrak{a}_K, \beta]$ is a simple stratum in A_K and*

$$Y(\beta, \mathfrak{a}) \cap G_K = Y(\beta, \mathfrak{a}_K),$$

where Y stands for any of $\boldsymbol{J}$, J^0, J^1, H^1.

(2) *Let $\theta \in \mathcal{C}(\mathfrak{a}, \beta, \psi_F)$, and set $\psi_K = \psi_F \circ \mathrm{Tr}_{K/F}$. The character*

$$\theta_K = \theta|_{H^1(\beta, \mathfrak{a}_K)} \tag{2.3.2}$$

lies in $\mathcal{C}(\mathfrak{a}_K, \beta, \psi_K)$.

We can work in the opposite direction, starting with a finite, tamely ramified field extension K/F and a non-trivial endo-class $\varPhi \in \mathcal{E}(K)$. We choose a realization φ of $\varPhi$, say $\varphi \in \mathcal{C}(\mathfrak{b}, \gamma, \psi_K)$ where $[\mathfrak{b}, \gamma]$ is a simple stratum in $B = \mathrm{End}_K(V)$ and V is a finite-dimensional K-vector space. Set $A = \mathrm{End}_F(V)$. There is then a unique hereditary $\mathfrak{o}_F$-order $\mathfrak{a}$ in A, stable under conjugation by $K^\times$ and such that $\mathfrak{b} = \mathfrak{a} \cap B$.

(2.3.3)

(1) *There exists a simple stratum $[\mathfrak{b}, \gamma']$ in B such that*

(a) $\mathcal{C}(\mathfrak{b}, \gamma', \psi_K) = \mathcal{C}(\mathfrak{b}, \gamma, \psi_K)$ *and*

(b) $[\mathfrak{a}, \gamma']$ *is a simple stratum in A.*

(2) *There exists a unique simple character $\varphi^F \in \mathcal{C}(\mathfrak{a}, \gamma', \psi_F)$ such that*

$$\varphi^F|_{H^1(\gamma', \mathfrak{b})} = \varphi.$$

(3) *The endo-class $\varPhi^F = cl(\varphi^F)$ depends only on the endo-class $\varPhi$ of φ. In particular, it does not depend on the choices of γ' and ψ_F.*

(4) *The map*

$$\begin{aligned} \mathfrak{i}_{K/F} : \mathcal{E}(K) &\longrightarrow \mathcal{E}(F), \\ \varPhi &\longmapsto \varPhi^F, \end{aligned} \tag{2.3.4}$$

is surjective with finite fibres. When K/F is Galois, the fibres of $\mathfrak{i}_{K/F}$ are also the orbits of $\mathrm{Gal}(K/F)$ in $\mathcal{E}(K)$.

(5) *If $F \subset K' \subset K$, then $\mathfrak{i}_{K/F} = \mathfrak{i}_{K'/F} \circ \mathfrak{i}_{K/K'}$.*

These assertions combine Theorem 7.10 and Corollary 9.13 of [3]. The relation in (3) is actually sharper, in that the character φ determines the character φ^F, as in [3] 7.15.

Let $\varTheta \in \mathcal{E}(F)$. The elements of the finite set $\mathfrak{i}_{K/F}^{-1}\varTheta$ are called the *K/F-lifts of $\varTheta$*. The only K/F-lift of $\mathbf{0}_F$ is $\mathbf{0}_K$. One deduces from [3] (11.2) the following method of computing the basic invariants (2.2.2) of the lifts of an endo-class.

(2.3.5) *Let $\Theta \in \mathcal{E}(F)$ have parameter field P/F. Let K/F be a finite, tamely ramified field extension. Write $K \otimes_F P = P_1 \times P_2 \times \cdots \times P_r$, where each P_i/K is a field extension. There is a canonical bijection*

$$\begin{aligned} \{P_1, P_2, \dots, P_r\} &\longrightarrow \mathfrak{i}_{K/F}^{-1}\Theta, \\ P_i &\longmapsto \Theta_i, \end{aligned}$$

such that P_i/K is a parameter field for Θ_i. This bijection is natural with respect to isomorphisms $K/F \to K'/F'$ of field extensions.

In the context of (2.3.1), the field $K[\beta]$ is K-isomorphic to one, and only one, of the simple components P_i of $K \otimes_F F[\beta]$. The endo-class, in $\mathcal{E}(K)$, of the character θ_K of (2.3.2) is the K/F-lift of $cl(\theta)$ corresponding to the component P_i.

2.4. Let $\Theta \in \mathcal{E}(F)$. We say that Θ is *totally wild* if $\deg \Theta = e(\Theta) = p^r$, for an integer $r \geqslant 0$. Equivalently, Θ is totally wild if any parameter field for Θ is totally wildly ramified over F.

Proposition. *Let $\Theta \in \mathcal{E}(F)$ have a parameter field P/F, and let T/F be the maximal tamely ramified sub-extension of P/F. Let K/F be a finite, tamely ramified field extension. The following conditions are equivalent:*

(1) *Θ has a totally wild K/F-lift;*
(2) *there exists an F-embedding $T \to K$.*

If P'/F is a parameter field for Θ and T'/F is the maximal tamely ramified sub-extension of P'/F, then T' is F-isomorphic to T.

Proof. All assertions follow readily from (2.3.5). □

We therefore define a tame parameter field for $\Theta \in \mathcal{E}(F)$ to be a finite tame extension E/F such that Θ has a totally wild E/F-lift, and such that E is minimal for this property. The proposition shows that Θ has a tame parameter field, that it is uniquely determined up to F-isomorphism (unlike the general parameter field).

The group $\mathrm{Aut}(T|F)$ acts on $\mathcal{E}(T)$. Using (2.3.5) again, the proposition yields:

Corollary. *Let $\Theta \in \mathcal{E}(F)$ have tame parameter field T/F. Let $\Psi \in \mathcal{E}(T)$ be a totally wild T/F-lift of Θ. The map $\gamma \mapsto \Psi^\gamma$ is a bijection between* $\mathrm{Aut}(T|F)$ *and the set of totally wild T/F-lifts of Θ.*

2.5. Some new terminology will be convenient. Let $[\mathfrak{a}, \beta]$ be a simple stratum in $A = \mathrm{M}_n(F)$; we say that $[\mathfrak{a}, \beta]$ is *m-simple* if $\mathfrak{a}$ is maximal among $\mathfrak{o}_F$-orders in A stable under conjugation by $F[\beta]^\times$.

Lemma 1. *Let $[\mathfrak{a}, \beta]$ be a simple stratum in A. The following conditions are equivalent.*

1. *$[\mathfrak{a}, \beta]$ is m-simple;*
2. *the order $\mathfrak{a}$ is principal and the ramification index $e(F[\beta]|F)$ is equal to the $\mathfrak{o}_F$-period of $\mathfrak{a}$;*
3. *the hereditary $\mathfrak{o}_{F[\beta]}$-order $\mathfrak{a}_{F[\beta]} = \mathfrak{a} \cap A_{F[\beta]}$ is maximal.*

Proof. See [17] 1.2.4. $\square$

A simple character θ is deemed m-simple if $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ where $[\mathfrak{a}, \beta]$ is m-simple. If $\Theta \in \mathcal{E}(F)$, an m-realization of Θ is a realization which is m-simple.

Lemma 2. *Two m-simple characters in $G = \mathrm{GL}_n(F)$ are endo-equivalent if and only if they are conjugate in G.*

Proof. This is a special case of (2.2.4). $\square$

Let $[\mathfrak{a}, \beta]$ be an m-simple stratum in A and set $P = F[\beta]$. By condition (3) of Lemma 1, the order $\mathfrak{a}_P$ is isomorphic to $\mathrm{M}_s(\mathfrak{o}_P)$, where $s = n/[P{:}F]$. We further have

$$\begin{aligned} \boldsymbol{J}(\beta, \mathfrak{a}) &= P^\times J^0(\beta, \mathfrak{a}), \\ J^0(\beta, \mathfrak{a})/J^1(\beta, \mathfrak{a}) &\cong \mathrm{GL}_s(\Bbbk_P). \end{aligned} \tag{2.5.1}$$

2.6. We need a more concrete version of the notion of tame parameter field.

Let θ be a simple character in $\mathrm{GL}_n(F)$. A *tame parameter field* for θ is a subfield T/F of $\mathrm{M}_n(F)$ of the following form: there is a simple stratum $[\mathfrak{a}, \beta]$ in $\mathrm{M}_n(F)$ such that $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ and T/F is the maximal tamely ramified sub-extension of $F[\beta]/F$. The proposition of 2.4 implies that a tame parameter field for θ is also a tame parameter field for $cl(\theta)$, and so is uniquely determined up to F-isomorphism. There is a more precise version of this uniqueness property.

Proposition. *Let θ be an m-simple character in $G = \mathrm{GL}_n(F)$. Let T_1/F, T_2/F be tame parameter fields for θ.*

(1) *There exists $j \in J^1_\theta$ such that $T_2 = T_1^j$.*
(2) *An element $y \in J^1_\theta$ normalizes $T_1^\times$ if and only if it centralizes $T_1^\times$.*

In particular, there is a unique F-isomorphism $T_1 \to T_2$ implemented by conjugation by an element of J^1_θ.

Proof. Before starting the proof, it will be useful to recall a standard technical result proved in the same way as, for example, [3] 15.19.

Conjugacy Lemma. *Let $\mathcal{G}$ be a pro p-group and let α be an automorphism of $\mathcal{G}$, of finite order relatively prime to p. Suppose $\mathcal{G}$ admits a descending chain $\{\mathcal{G}_i\}_{i\geqslant 1}$ of open, normal, α-stable subgroups $\mathcal{G}_i$ such that $\mathcal{G}_i/\mathcal{G}_{i+1}$ is a finite abelian p-group and $[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j}$, for all $i, j \geqslant 1$. In the group $\langle\alpha\rangle \ltimes \mathcal{G}$, any element αg, $g \in \mathcal{G}$, is $\mathcal{G}$-conjugate to one of the form αh, where $h \in \mathcal{G}$ commutes with α.*

We prove the proposition. We choose a simple stratum $[\mathfrak{a}, \beta_i]$ in $A = \mathrm{M}_n(F)$ such that $\theta \in \mathcal{C}(\mathfrak{a}, \beta_i,)$ and $T_i \subset P_i = F[\beta_i]$. Thus $J^1_\theta = J^1(\beta_i, \mathfrak{a})$, and so on. We abbreviate $J^1 = J^1_\theta$.

Let K_i/F be the maximal unramified sub-extension of T_i/F. Consider the ring $\mathfrak{J} = \mathfrak{J}(\beta_i, \mathfrak{a})$ (in the notation of [17] 3.1.8). If we write $\mathfrak{J}^1 = \mathfrak{J} \cap \mathfrak{p}_\mathfrak{a}$, then $J^1 = 1{+}\mathfrak{J}^1$ and the ring $\mathfrak{j} = \mathfrak{J}/\mathfrak{J}^1$ is isomorphic to $\mathrm{M}_s(\Bbbk_{P_i})$, where $s[P_i{:}F] = n$. The centre $\mathfrak{z}$ of $\mathfrak{j}^\times$ is therefore $\Bbbk_{P_i}^\times$. Since $\mathfrak{o}_{P_i} \cap \mathfrak{J}^1 = \mathfrak{p}_{P_i}$, reduction modulo $\mathfrak{p}_{P_i}$ induces an isomorphism $\boldsymbol{\mu}_{P_i} \cong \mathfrak{z}$. Since P_i/K_i is totally ramified and $\mathfrak{J}^1 \cap \mathfrak{o}_{K_i} = \mathfrak{p}_{K_i}$, the group $\boldsymbol{\mu}_{K_i}$ also maps isomorphically to $\mathfrak{z}$ under reduction modulo $\mathfrak{J}^1$. So, if ζ_1 is a generator of $\boldsymbol{\mu}_{K_1}$, there exists $\zeta_2 \in \boldsymbol{\mu}_{K_2}$ such that $\zeta_2 = \zeta_1 j$, for some $j \in J^1$. The roots of unity ζ_1, ζ_2 have the same order, so ζ_2 generates $\boldsymbol{\mu}_{K_2}$.

We apply the Conjugacy Lemma to $\mathcal{G} = J^1$ (with its standard filtration) and the automorphism α induced by conjugation by ζ_1. We deduce that $\zeta_2 = \zeta_1 j$ is J^1-conjugate to an element $\zeta_3 = \zeta_1 j_1$, where $j_1 \in J^1$ commutes with ζ_1. Thus $j_1 = \zeta_1^{-1}\zeta_3$ has finite order dividing that of ζ_1, and this order is not divisible by p. Since J^1 is a pro-p group, we deduce that $j_1 = 1$.

Therefore, after applying a J^1-conjugation to T_2 (which does not affect the

conditions imposed on T_2), we may assume that $K_1 = K_2 = K$, say. Let A_K be the centralizer of K in $\mathrm{M}_n(F)$, set $G_K = A_K^\times$, and put $\mathfrak{a}_K = \mathfrak{a} \cap A_K$. As in (2.3.1), the pair $[\mathfrak{a}_K, \beta_i]$ is a simple stratum in A_K and $\mathcal{C}(\mathfrak{a}_K, \beta_1) = \mathcal{C}(\mathfrak{a}_K, \beta_2)$. Moreover, $\boldsymbol{J}(\beta_i, \mathfrak{a}_K) = \boldsymbol{J}(\beta_i, \mathfrak{a}) \cap G_K = \boldsymbol{J}_K$, say. Also, $J^1(\beta_i, \mathfrak{a}_K) = J^1 \cap G_K = J^1_K$, say.

The extensions T_1/K, T_2/K are isomorphic and totally tamely ramified. Let ϖ be a prime element of T_1 such that $\varpi^{[T_1:K]} = \varpi_K$, for some prime element ϖ_K of K. Since we are in the m-simple case, the centre of the group $\boldsymbol{J}_K/J^1_K$ is $P_i^\times J^1_K/J^1_K$. The groups $T_1^\times$, $T_2^\times$ have the same image in $\boldsymbol{J}(\beta, \mathfrak{a}_K)/J^1(\beta, \mathfrak{a}_K)$, and we can argue exactly as before to complete the proof of (1).

In part (2), write $T = T_1$ and take K as before. Let $j \in J^1$ normalize $T^\times$. Surely $\mathrm{Ad}\, j$ acts trivially on the group $\boldsymbol{\mu}_K = \boldsymbol{\mu}_T$, whence j centralizes $K^\times$. That is, $j \in J^1_K = J^1(\beta, \mathfrak{a}_K)$. Taking a prime element ϖ as before, $j\varpi j^{-1} = \zeta\varpi$, for some $\zeta \in \boldsymbol{\mu}_K$. However, $j\varpi j^{-1} \equiv \varpi \pmod{J^1_K}$, whence $\zeta = 1$, as required. $\square$

The condition of m-simplicity imposed on θ is unnecessary. However, it shortens the arguments a little and is the only case we need.

Remark. In the situation of the proposition, there may be a field extension T_0/F, with $T_0^\times \subset \boldsymbol{J}_\theta$, which is isomorphic to a tame parameter field for θ without being one. See 10.1 Remark for a context in which such examples arise.

2.7. We connect our two uses of the phrase "tame parameter field". As a matter of notation, if we have an isomorphism $\boldsymbol{\iota} : F \to F'$ of fields, we denote by $\boldsymbol{\iota}_*$ the induced bijection $\mathcal{E}(F) \to \mathcal{E}(F')$.

Proposition. *Let $\Theta \in \mathcal{E}(F)$ have parameter field E/F, and let Ψ be a totally wild E/F-lift of Θ. Let $d = \deg \Theta$ and $n = rd$, for an integer $r \geqslant 1$. Let $\boldsymbol{\iota}$ be an F-embedding of E in $A = \mathrm{M}_n(F)$. There exists an m-simple character θ in $G = \mathrm{GL}_n(F)$, with the following properties:*

(a) *$cl(\theta) = \Theta$;*
(b) *the field $\boldsymbol{\iota}E$ is a tame parameter field for θ;*
(c) *if $\theta_{\boldsymbol{\iota}E}$ is defined as in* (2.3.2), *then $cl(\theta_{\boldsymbol{\iota}E}) = \boldsymbol{\iota}_*\Psi$.*

These conditions determine θ uniquely, up to conjugation by an element of the G-centralizer $G_{\boldsymbol{\iota}E}$ of $\boldsymbol{\iota}E^\times$.

Proof. Let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ be an m-realization of Θ, and let T/F be the maximal

tamely ramified sub-extension of $F[\beta]/F$. The fields T, $\boldsymbol{\iota}E$ are F-isomorphic and hence G-conjugate. So, replacing θ by a G-conjugate, we can assume $T = \boldsymbol{\iota}E$. The endo-class $cl(\theta_T)$ is then a totally wild lift of Θ. Let $g \in G$ normalize $\boldsymbol{\iota}E$. We have the relation $cl(\theta^g_{\boldsymbol{\iota}E}) = cl(\theta_{\boldsymbol{\iota}E})^g$. Invoking 2.4 Corollary, we may choose θ to achieve $cl(\theta_{\boldsymbol{\iota}E}) = \boldsymbol{\iota}_*\Psi$. This character θ satisfies the conditions (a)–(c).

For the uniqueness assertion, let θ, θ' be m-simple characters in G satisfying conditions (a)–(c) relative to the field $\boldsymbol{\iota}E$. Condition (c) says that the m-simple characters $\theta_{\boldsymbol{\iota}E}$, $\theta'_{\boldsymbol{\iota}E}$ in $G_{\boldsymbol{\iota}E}$ are endo-equivalent over $\boldsymbol{\iota}E$, and so are $G_{\boldsymbol{\iota}E}$-conjugate (2.5 Lemma 2). We may as well assume, therefore, that $\theta_{\boldsymbol{\iota}E} = \theta'_{\boldsymbol{\iota}E}$. Condition (a) implies, via 2.5 Lemma 2, that $\theta' = \theta^g$, for some $g \in G$. The element g conjugates $\boldsymbol{\iota}E$ to a tame parameter field for θ'. Replacing g by gj, for some $j \in J^1_{\theta'}$, we may assume that $(\boldsymbol{\iota}E)^g = \boldsymbol{\iota}E$ (2.6 Proposition). In other words, g normalizes $\boldsymbol{\iota}E$. Since θ' satisfies (c), 2.4 Corollary implies that g commutes with $\boldsymbol{\iota}E$, as required. □

Corollary. *For $k = 1, 2$, let $\boldsymbol{\iota}^{(k)}$ be an F-embedding of E in A and let $\theta^{(k)}$ be an m-simple character in G, of endo-class Θ and for which $\boldsymbol{\iota}^{(k)}E/F$ is a tame parameter field. Let Φ_k denote the endo-class of $\theta^{(k)}_{\boldsymbol{\iota}^{(k)}E}$. The pairs $(\boldsymbol{\iota}^{(k)}, \theta^{(k)})$ are conjugate in G if and only if*

$$(\boldsymbol{\iota}^{(1)}_*)^{-1}\,\Phi_1 = (\boldsymbol{\iota}^{(2)}_*)^{-1}\,\Phi_2.$$

Proof. Replacing one pair by a G-conjugate, we can assume $\boldsymbol{\iota}^{(1)} = \boldsymbol{\iota}^{(2)} = \boldsymbol{\iota}$ say. By the proposition, the characters $\theta^{(j)}$ give rise to the same endo-class over E if and only if they are $G_{\boldsymbol{\iota}E}$-conjugate. □

3. Action of tame characters

We take a (possibly trivial) m-simple character θ in $G = \mathrm{GL}_n(F)$, and let T/F be a tame parameter field for θ. We analyze various families of irreducible representations of $\boldsymbol{J}_\theta$ which contain θ and show that these families carry canonical actions of the group $X_1(T)$. We follow throughout the notational conventions introduced in 2.3.

3.1. Let θ be an m-simple character in $G = \mathrm{GL}_n(F)$ and let $I_G(\theta)$ denote the set of elements of G which intertwine θ.

Definition. *A character of* $\boldsymbol{J}_\theta$ *is* θ-flat *if it is trivial on* J^1_θ *and is intertwined by every element of* $I_G(\theta)$.

For example, if $\chi \in X_1(F)$, then $\chi \circ \det|_{\boldsymbol{J}_\theta}$ is a θ-flat character of $\boldsymbol{J}_\theta$. Let $X_1(\theta)$ denote the group of θ-flat characters of $\boldsymbol{J}_\theta$.

Proposition. *Let* T/F *be a tame parameter field for* θ *and let* $\det_T : G_T \to T^\times$ *be the determinant map.*

(1) *Let* $\phi \in X_1(T)$. *There is a unique* θ*-flat character* $\phi^{\boldsymbol{J}}$ *of* $\boldsymbol{J} = \boldsymbol{J}_\theta$ *such that* $\phi^{\boldsymbol{J}}(x) = \phi(\det_T x)$, *for all* $x \in \boldsymbol{J} \cap G_T$.

(2) *The map*

$$\begin{aligned} X_1(T) &\longrightarrow X_1(\theta), \\ \phi &\longmapsto \phi^{\boldsymbol{J}}, \end{aligned} \tag{3.1.1}$$

is a surjective homomorphism of abelian groups. Its kernel is the group $X_0(T)_s$ *of unramified characters* χ *of* $T^\times$ *satisfying* $\chi^s = 1$, *where* $s = n/\deg cl(\theta)$.

(3) *Let* T'/F *be a tame parameter field for* θ. *For* $\xi \in X_1(T')$, *define* $\xi^{\boldsymbol{J}} \in X_1(\theta)$ *as in* (1), *using* T' *in place of* T. *If* $j \in J^1_\theta$ *satisfies* $T' = T^j$, *then*

$$\left(\phi^j\right)^{\boldsymbol{J}} = \phi^{\boldsymbol{J}}, \quad \phi \in X_1(T).$$

Proof. By definition, there is a simple stratum $[\mathfrak{a}, \beta]$ in A such that $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ and $T \subset F[\beta]$. Set $P = F[\beta]$, and use the standard notation of 2.3. Thus $\boldsymbol{J}_\theta = P^\times U_{\mathfrak{a}_P} J^1_\theta$ (2.5.1), and $P^\times U_{\mathfrak{a}_P} \cap J^1_\theta = P^\times U_{\mathfrak{a}_P} \cap U^1_{\mathfrak{a}} = U^1_{\mathfrak{a}_P}$. Let $\det_P : G_P \to P^\times$ denote the determinant map.

Let $\chi \in X_1(P)$ and define a character $\tilde\chi$ of $\boldsymbol{J}_\theta$ by

$$\tilde\chi(xj) = \chi(\det_P x), \quad x \in P^\times U_{\mathfrak{a}_P},\ j \in J^1_\theta.$$

We show that $\tilde\chi \in X_1(\theta)$. Surely $\tilde\chi$ is trivial on J^1_θ. The set $I_G(\theta)$ is $J^1_\theta G_P J^1_\theta$. If $y \in G_P$, then $y^{-1}\boldsymbol{J}_\theta y \cap \boldsymbol{J}_\theta = P^\times(y^{-1}J^0_\theta y \cap J^0_\theta)$. The characters $\tilde\chi$, $\tilde\chi^y$ agree on $P^\times$, and we are so reduced to showing that y intertwines $\tilde\chi|_{J^0_\theta}$. This will follow from:

Lemma. *Set $K = U_{\mathfrak{a}_P} U^1_{\mathfrak{a}}$, and define a character $\bar{\chi}$ of K by*

$$\bar{\chi}(hu) = \chi(\det_P h), \quad h \in U_{\mathfrak{a}_P},\ u \in U^1_{\mathfrak{a}}.$$

The character $\bar{\chi}$ is intertwined by every element y of G_P.

Proof. Let $y \in G_P$. We show

$$K \cap K^y = (U_{\mathfrak{a}_P} \cap U^y_{\mathfrak{a}_P})(U^1_{\mathfrak{a}} \cap U^{1\,y}_{\mathfrak{a}}). \tag{3.1.2}$$

The characters $\bar{\chi}$, $\bar{\chi}^y$ agree on each factor, so the lemma will follow. The relation (3.1.2) is equivalent to the additive relation

$$(\mathfrak{a}_P + \mathfrak{p}_{\mathfrak{a}}) \cap (\mathfrak{a}_P + \mathfrak{p}_{\mathfrak{a}})^y = \mathfrak{a}_P \cap \mathfrak{a}_P^y + \mathfrak{p}_{\mathfrak{a}} \cap \mathfrak{p}_{\mathfrak{a}}^y, \tag{3.1.3}$$

where $\mathfrak{p}_{\mathfrak{a}}$ is the Jacobson radical of $\mathfrak{a}$. Techniques to deal with such identities are developed in the early chapters of [17], but we give an *ad hoc* argument of the same kind. We view A_P as $\mathrm{End}_P(V)$, where V is a P-vector space of dimension s. Let L be an $\mathfrak{a}_P$-lattice in V. We choose an $\mathfrak{o}_P$-basis of L to identify A_P with $\mathrm{M}_s(P)$ and $\mathfrak{a}_P$ with $\mathrm{M}_s(\mathfrak{o}_P)$. Let $A(P) = \mathrm{End}_F(P)$ and let $\mathfrak{a}(P)$ be the unique hereditary $\mathfrak{o}_F$-order in $A(P)$ stable under conjugation by $P^\times$. The Jacobson radical of $\mathfrak{a}(P)$ is then $\varpi\mathfrak{a}(P)$, where ϖ is a prime element of $\mathfrak{o}_P$. The order $\mathfrak{a}$ is $\mathrm{M}_s(\mathfrak{a}(P))$, and its radical is $\varpi\mathfrak{a} = \mathrm{M}_s(\varpi\mathfrak{a}(P))$. The desired properties (3.1.2), (3.1.3) depend only on the double coset $U_{\mathfrak{a}_P} y U_{\mathfrak{a}_P}$. We may therefore take y diagonal, with all its eigenvalues powers of ϖ. The relation (3.1.3) follows immediately. $\square$

We have shown that $\chi \mapsto \tilde{\chi}$ gives a homomorphism $X_1(P) \to X_1(\theta)$. Further, $\tilde{\chi} = 1$ if and only if χ is trivial on $\det_P P^\times U_{\mathfrak{a}_P} = (P^\times)^s U_P$, that is, if and only if $\chi \in X_0(P)_s$.

On the other hand, if $\xi \in X_1(\theta)$, then the restriction of ξ to $\boldsymbol{J}_\theta \cap G_P = P^\times U_{\mathfrak{a}_P}$ is intertwined by every element of G_P. This implies that $\xi|_{\boldsymbol{J}_\theta \cap G_P}$ factors through $\det_P$ and so ξ lies in the image of $X_1(P)$. The map $X_1(P) \to X_1(\theta)$, $\chi \mapsto \tilde{\chi}$, is therefore surjective and induces an isomorphism $X_1(P)/X_0(P)_s \to X_1(\theta)$.

We next observe that $\boldsymbol{J}_\theta = (\boldsymbol{J}_\theta \cap G_T)\, J^1_\theta$, and $G_T \cap J^1_\theta \subset U^1_{\mathfrak{a}_T}$. The field extension P/T is totally wildly ramified, so the map $\phi \mapsto \phi \circ \mathrm{N}_{P/T}$ gives an isomorphism $X_1(T) \to X_1(P)$ carrying $X_0(T)_s$ to $X_0(P)_s$. For $\phi \in X_1(T)$,

the character $(\phi\circ \mathrm{N}_{P/T})^{\sim}$ of $\boldsymbol{J}_\theta$ satisfies the defining property for $\phi^{\boldsymbol{J}}$, and we have proven the first two parts of the proposition. Part (3) is an immediate consequence of the definitions. $\square$

Remark. The identification of $X_1(\theta)$ with a quotient of $X_1(P)$ (as in the proof of the proposition) is convenient for calculation. It cannot be regarded as canonical, since P is not unambiguously defined by θ. However, the proposition shows that the identification of $X_1(\theta)$ with $X_1(T)/X_0(T)_s$ is canonical.

3.2. We continue with the m-simple character θ of 3.1. Let η be the unique irreducible representation of J^1_θ which contains θ, as in [17] (5.1.1). We refer to η as the 1-*Heisenberg representation over θ.*

Definition. *A* full Heisenberg representation over θ *is a representation κ of $\boldsymbol{J}_\theta$ such that*

(a) $\kappa|_{J^1_\theta}\cong\eta$ *and*
(b) κ *is intertwined by every element of* $I_G(\theta)$*.*

We denote by $\mathcal{H}(\theta)$ the set of equivalence classes of full Heisenberg representations over θ. We observe that, if $\kappa\in\mathcal{H}(\theta)$ and $\phi\in X_1(\theta)$, then $\phi\otimes\kappa$ also lies in $\mathcal{H}(\theta)$. Thus $\mathcal{H}(\theta)$ comes equipped with a canonical action of $X_1(\theta)$.

Proposition. *Let θ be an m-simple character in $G=\mathrm{GL}_n(F)$.*

(1) *There exists a full Heisenberg representation over θ.*
(2) *If $\kappa\in\mathcal{H}(\theta)$ and $\phi\in X_1(\theta)$, then $\phi\otimes\kappa\in\mathcal{H}(\theta)$ and the pairing*

$$\begin{aligned} X_1(\theta)\times\mathcal{H}(\theta)&\longrightarrow\mathcal{H}(\theta),\\ (\phi,\kappa)&\longmapsto\phi\otimes\kappa, \end{aligned}$$

endows $\mathcal{H}(\theta)$ with the structure of principal homogeneous space over $X_1(\theta)$.

Proof. The proof of part (1) requires a different family of techniques which we do not use elsewhere. We therefore give it separate treatment in 3.3 below. Here we deduce part (2).

We need an intermediate step. Let $[\mathfrak{a},\beta]$ be a simple stratum such that $\theta\in\mathcal{C}(\mathfrak{a},\beta)$, and set $P=F[\beta]$. We abbreviate $J^1=J^1_\theta$, and so on. Thus

$I_G(\theta) = J^1 G_P J^1$. Let $\mathcal{H}^0(\theta)$ be the set of equivalence classes of representations μ of J^0 satisfying $\mu|_{J^1} \cong \eta$ and such that $I_G(\mu) = I_G(\theta)$: in the language of [17], the elements of $\mathcal{H}^0(\theta)$ are the "β-extensions of η".

Let $X_1^0(\theta)$ denote the group of characters of J^0 which are trivial on J^1 and are intertwined by every element of $I_G(\theta)$.

Lemma.

(1) *Let $\chi \in X_1^0(\theta)$. The restriction of χ to $U_{\mathfrak{a}_P} = J^0 \cap G_P$ factors through $\det_P$, and χ is the restriction to J^0 of some $\tilde{\chi} \in X_1(\theta)$.*

(2) *The set $\mathcal{H}^0(\theta)$ is a principal homogeneous space over $X_1^0(\theta)$. Distinct elements of $\mathcal{H}^0(\theta)$ do not intertwine in G.*

Proof. In (1), the restriction of χ to $U_{\mathfrak{a}_P}$ is intertwined by every element of G_P, and so must factor through $\det_P$. The second assertion follows from the description of elements of $X_1(\theta)$ given in the proof of 3.1 Proposition. Part (2) is part (iii) of [17] (5.2.2). □

Let $\kappa, \kappa' \in \mathcal{H}(\theta)$. By the lemma, there exists $\chi \in X_1(\theta)$ such that $\chi \otimes \kappa$ agrees with κ' on J^0. Thus there exists a character ϕ of $\boldsymbol{J}$, trivial on J^0, such that $\phi\chi \otimes \kappa \cong \kappa'$. The character $\phi\chi$ lies in $X_1(\theta)$, so the set $\mathcal{H}(\theta)$ comprises a single $X_1(\theta)$-orbit.

Next, let $\kappa \in \mathcal{H}(\theta)$, $\chi \in X_1(\theta)$, and suppose that $\kappa \cong \chi \otimes \kappa$. The lemma implies that χ is trivial on J^0. We view κ and $\chi \otimes \kappa$ as acting on the same vector space V, and that $\kappa|_{J^1} = \eta$. By hypothesis, there is an automorphism f of V such that $\chi(x)\kappa(x) \circ f = f \circ \kappa(x)$, for $x \in \boldsymbol{J}$. In particular, $\eta(y) \circ f = f \circ \eta(y)$, $y \in J^1$. Thus f is a scalar, whence $\chi = 1$, as required.

We have shown that the set $\mathcal{H}(\theta)$, if non-empty, is a principal homogeneous space over $X_1(\theta)$. □

Using 3.1 Proposition, we can equally view $\mathcal{H}(\theta)$ as $X_1(T)$-space. We use the notation

$$\xi \odot \kappa = \xi^{\boldsymbol{J}} \otimes \kappa, \tag{3.2.1}$$

for $\xi \in X_1(T)$ and $\kappa \in \mathcal{H}(\theta)$.

Corollary. *With respect to the action* (3.2.1), *the set* $\mathcal{H}(\theta)$ *is a principal homogeneous space over* $X_1(T)/X_0(T)_s$, *where* $s = n/\deg cl(\theta)$.

Proof. The corollary follows directly from the proposition and 3.1 Proposition. □

Remark. Let τ be an irreducible representation of $\boldsymbol{J}_\theta$ such that $\tau|_{H^1_\theta}$ contains θ. The proof of the proposition shows that $\tau \in \mathcal{H}(\theta)$ if and only if $\tau|_{J^0_\theta} \in \mathcal{H}^0(\theta)$.

3.3. We prove part (1) of 3.2 Proposition. Specifically, we are given an m-simple stratum $[\mathfrak{a}, \beta]$ in $A = \mathrm{M}_n(F)$ and a simple character $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$. Let η be the 1-Heisenberg representation over θ. Writing $G = \mathrm{GL}_n(F)$ and $P = F[\beta]$, so that G_P is the G-centralizer of $P^\times$, we have $I_G(\theta) = J^0 G_P J^0$. We have to prove:

Proposition. *There exists a representation* κ *of* $\boldsymbol{J}_\theta$ *such that* $\kappa|_{J^1_\theta} \cong \eta$ *and which is intertwined by every element of* G_P.

Proof. As recalled in 3.2, there exists a representation μ of $J^0 = J^0_\theta$, extending η and which is intertwined by every element of G_P. This surely admits extension to a representation of $\boldsymbol{J} = P^\times J^0$. If $[P{:}F] = n$, then $G_P = P^\times$ and there is nothing to prove.

To deal with the general case, we first recall from [17] and [3] one method of constructing the representation μ. We abbreviate $\mathfrak{b} = \mathfrak{a}_P$ and let L be a $\mathfrak{b}$-lattice in $V = F^n$. If we choose a prime element ϖ of P, then $\{\varpi^j L : j \in \mathbb{Z}\}$ is the $\mathfrak{o}_F$-lattice chain in V defining the hereditary order $\mathfrak{a}$. Let $s = n/[P{:}F]$. We choose a decomposition

$$L = L_0 \oplus L_0 \oplus \ \cdots \ \oplus L_0$$

of L, in which L_0 is an $\mathfrak{o}_P$-lattice of rank one. We accordingly write

$$V = V_0 \oplus V_0 \oplus \ \cdots \ \oplus V_0.$$

Let M be the subgroup of G preserving this decomposition of V. In particular, M is a Levi component of a parabolic subgroup Q_+ of G. Let Q_- be the M-opposite of Q_+, and let $N_\pm$ be the unipotent radical of $Q_\pm$. The obvious projections $V \to V_0$ lie in $\mathfrak{a}$, so we get an Iwahori decomposition

$$U^1_\mathfrak{a} = U^1_\mathfrak{a} \cap N_- \ \cdot \ U^1_\mathfrak{a} \cap M \ \cdot \ U^1_\mathfrak{a} \cap N_+. \tag{3.3.1}$$

Moreover, $U^1_{\mathfrak{a}} \cap M$ is the direct product of s groups $U^1_{\mathfrak{a}_0}$, where $\mathfrak{a}_0$ is the hereditary $\mathfrak{o}_F$-order in $\mathrm{End}_F(V_0)$ defined by the lattice chain $\{\varpi^j L_0 : j \in \mathbb{Z}\}$. The pair $[\mathfrak{a}_0, \beta]$ is an m-simple stratum in $\mathrm{End}_F(V_0)$.

The group $U_{\mathfrak{a}}$ does not admit an Iwahori decomposition in general, but we do have

$$U_{\mathfrak{a}} \cap Q_+ = U_{\mathfrak{a}} \cap M \ \cdot \ U_{\mathfrak{a}} \cap N_+,$$
$$U_{\mathfrak{a}} \cap M = U_{\mathfrak{a}_0} \times U_{\mathfrak{a}_0} \times \ \cdots \ \times U_{\mathfrak{a}_0}.$$

Lemma 1. *Let* $H^1 = H^1_\theta = H^1(\beta, \mathfrak{a})$ *and similarly for* J^1, J^0.

(1) *There are Iwahori decompositions*

$$H^1 = H^1 \cap N_- \ \cdot \ H^1 \cap M \ \cdot \ H^1 \cap N_+,$$
$$H^1 \cap M = H^1(\beta, \mathfrak{a}_0) \times H^1(\beta, \mathfrak{a}_0) \times \ \cdots \ \times H^1(\beta, \mathfrak{a}_0),$$

and

$$J^1 = J^1 \cap N_- \ \cdot \ J^1 \cap M \ \cdot \ J^1 \cap N_+,$$
$$J^1 \cap M = J^1(\beta, \mathfrak{a}_0) \times J^1(\beta, \mathfrak{a}_0) \times \ \cdots \ \times J^1(\beta, \mathfrak{a}_0).$$

(2) *The characters* $\theta|_{H^1 \cap N_-}$, $\theta|_{H^1 \cap N_+}$ *are trivial, while there is a unique character* $\theta_0 \in \mathcal{C}(\mathfrak{a}_0, \beta)$ *such that*

$$\theta|_{H^1 \cap M} = \theta_0 \otimes \theta_0 \otimes \ \cdots \ \otimes \theta_0.$$

(3) *The set*

$$J^1_+ = H^1 \cap N_- \ \cdot \ J^1 \cap Q_+$$

is a group. Let η_0 *be the* 1*-Heisenberg representation of* $J^1(\beta, \mathfrak{a}_0)$ *over* θ_0. *There is a unique representation* η_+ *of* J^1_+, *which is trivial on* $H^1 \cap N_-$ *and* $J^1 \cap N_+$, *and which restricts to* $\eta_0 \otimes \eta_0 \otimes \cdots \otimes \eta_0$ *on* $J^1 \cap M$. *This representation satisfies*

$$\mathrm{Ind}_{J^1_+}^{J^1} \eta_+ \cong \eta.$$

Proof. This is an instance of the discussion in [3] §10, especially Example 10.9. □

Remark. The simple characters θ, θ_0 are endo-equivalent.

For the next step, we take $\mu_0 \in \mathcal{H}^0(\theta_0)$ (in the notation of 3.2). The group $H^1 \cap N_- \cdot J^0 \cap Q_+$ admits a unique representation μ_+ which is trivial on $H^1 \cap N_-$ and $J^0 \cap N_+$, and which restricts to $\mu_0 \otimes \mu_0 \otimes \cdots \otimes \mu_0$ on $J^0 \cap M$.

Lemma 2.

(1) *The representation $\tilde\mu_+$ of $J^1 \cap N_- \cdot J^0 \cap Q_+$, induced by μ_+, is irreducible, and satisfies $\tilde\mu_+|_{J^1} \cong \eta$.*
(2) *There is a unique representation μ of J^0 extending $\tilde\mu_+$.*
(3) *The representation μ lies in $\mathcal{H}^0(\theta)$.*

Proof. Assertion (1) is straightforward. The restriction of $\tilde\mu_+$ to $H^1 \cap N_- \cdot J^1 \cap M \cdot J^0 \cap N_+$ is the representation denoted $\tilde\eta_{\mathrm{M}}$ in [17] (5.1.14). It admits extension to a representation μ' of J^0, and any such extension lies in $\mathcal{H}^0(\theta)$ [17] (5.2.4). Inspection of intertwining implies that $\mu'|_{J^0\cap M}$ is of the form $\mu_0' \otimes \cdots \otimes \mu_0'$, for some $\mu_0' \in \mathcal{H}^0(\theta_0)$. By 3.2 Lemma, we may choose μ' such that $\mu_0' \cong \mu_0$. This proves (2) and (3). □

As remarked at the beginning of the proof, there exists $\kappa_0 \in \mathcal{H}(\theta_0)$ such that $\kappa_0|_{J^0(\beta,\mathfrak{a}_0)} \cong \mu_0$. We use κ_0 to define a representation κ_+ of the group $H^1 \cap N_- \cdot P^\times J^0 \cap Q_+$, extending μ_+ and agreeing with $\kappa_0 \otimes \cdots \otimes \kappa_0$ on $P^\times J^0 \cap M$. We then induce κ_+ to get a representation $\tilde\kappa_+$ of $J^1 \cap N_- \cdot P^\times J^0 \cap Q_+$ extending $\tilde\mu_+$.

The representation μ of Lemma 2 is stable under conjugation by $P^\times$ and so admits extension to $\boldsymbol{J}$. This extension is uniquely determined up to tensoring with a character of $\boldsymbol{J}/J^0 \cong P^\times/U_P$. We may therefore choose this extension, call it κ, to agree with $\tilde\kappa_+$ on $J^1 \cap N_- \cdot P^\times J^0 \cap Q_+$.

To complete the proof, we have to show that κ is intertwined by any element y of $I_G(\theta) = J^0 G_P J^0$. It will only be the coset $J^0 y J^0$ which intervenes, so we may assume $y \in G_P$. Indeed, $J^0 G_P J^0 = J^0 (G_P \cap M) J^0$, so we take $y \in G_P \cap M$.

The element y intertwines η *with multiplicity one,* in the following sense: there is a non-zero $J^1 \cap {J^1}^y$-homomorphism $\phi : \eta \to \eta^y$ and ϕ is uniquely determined,

up to scalar multiple [17] (5.1.8). The element y also intertwines μ, by definition, and it therefore does so with multiplicity one. That is, ϕ provides a $J^0 \cap J^{0y}$-homomorphism $\mu \to \mu^y$. Surely y intertwines κ_+, hence also $\tilde{\kappa}_+$. That implies ϕ to be a homomorphism relative to the group $(J^1 \cap N_- \cdot P^\times J^0 \cap Q_+) \cap (J^1 \cap N_- \cdot P^\times J^0 \cap Q_+)^y$. So, ϕ is a homomorphism relative to the group generated by

$$J^0 \cap J^{0y} \cup \big((J^1 \cap N_- \cdot P^\times J^0 \cap Q_+) \cap (J^1 \cap N_- \cdot P^\times J^0 \cap Q_+)^y\big).$$

This surely contains $(J^0 \cap J^{0y})P^\times = \boldsymbol{J} \cap \boldsymbol{J}^y$. We have shown that y intertwines κ, as required. $\square$

The proof of the proposition yields rather more. First, it produces a map

$$\begin{aligned} f_s : \mathcal{H}(\theta_0) &\longrightarrow \mathcal{H}(\theta), \\ \kappa_0 &\longmapsto \kappa, \end{aligned} \tag{3.3.2}$$

using the same notation as in the proof. If T/F is the maximal tamely ramified sub-extension of P/F, then $\mathcal{H}(\theta_0)$ is a principal homogeneous space over $X_1(T)$ and $\mathcal{H}(\theta)$ is a principal homogeneous space over $X_1(T)/X_0(T)_s$. A simple check yields:

Corollary 1. *Let T/F be the maximal tamely ramified sub-extension of P/F. The map* (3.3.2) *satisfies*

$$f_s(\phi \odot \kappa_0) = \phi \odot f_s(\kappa_0), \quad \kappa_0 \in \mathcal{H}(\theta_0),\ \phi \in X_1(T).$$

It is surjective, and its fibres are the orbits of $X_0(T)_s$ in $\mathcal{H}(\theta_0)$.

The intertwining analysis in the proof further implies:

Corollary 2. *Distinct elements of $\mathcal{H}(\theta)$ do not intertwine in G.*

3.4. We recall some fundamental results from [17], re-arranged to suit our present purposes. Let $\boldsymbol{J}$ be an open, compact modulo centre, subgroup of G and Λ an irreducible smooth representation of $\boldsymbol{J}$.

Definition. *The pair $(\boldsymbol{J}, \Lambda)$ is an* extended maximal simple type *in G if there is an m-simple character θ in G such that*

1. $\boldsymbol{J} = \boldsymbol{J}_\theta$,
2. *the pair $(J^0_\theta, \Lambda|_{J^0_\theta})$ is a maximal simple type in G, and*
3. *$\Lambda|_{H^1_\theta}$ is a multiple of θ.*

For the phrase "maximal simple type", see 5.5.10 and 6.2 of [17] or 3.6 below.

Remark. Let θ be an m-simple character in $\mathrm{GL}_n(F)$. An irreducible representation τ of $\boldsymbol{J}_\theta$ containing θ is an extended maximal simple type over θ if and only if $I_G(\tau) = \boldsymbol{J}_\theta$.

If θ is an m-simple character in G, we denote by $\mathcal{T}(\theta)$ the set of equivalence classes of extended maximal simple types $(\boldsymbol{J}_\theta, \Lambda)$ with $\Lambda|_{H^1_\theta}$ a multiple of θ. We recall [17] (6.1.2) that distinct elements of $\mathcal{T}(\theta)$ do not intertwine in G.

Let P/F be a parameter field for θ and T/F the tame parameter field it contains. Let $\Lambda \in \mathcal{T}(\theta)$. If $\phi \in X_1(T)$ we define $\phi^{\boldsymbol{J}} \in X_1(\theta)$ as in 3.1, and form the representation

$$\phi \odot \Lambda = \phi^{\boldsymbol{J}} \otimes \Lambda. \tag{3.4.1}$$

Proposition. *Let θ be an m-simple character in $G = \mathrm{GL}_n(F)$, and let T/F be a tame parameter field for θ.*

(1) *If $\phi \in X_1(T)$ and $\Lambda \in \mathcal{T}(\theta)$, the representation $\phi \odot \Lambda$ of* (3.4.1) *lies in $\mathcal{T}(\theta)$. The canonical pairing*

$$\begin{aligned} X_1(T) \times \mathcal{T}(\theta) &\longrightarrow \mathcal{T}(\theta), \\ (\phi, \Lambda) &\longmapsto \phi \odot \Lambda. \end{aligned} \tag{3.4.2}$$

defines a structure of $X_1(T)$-space on the set $\mathcal{T}(\theta)$.

(2) *Suppose that* $\deg cl(\theta) = n$. *The sets $\mathcal{H}(\theta)$, $\mathcal{T}(\theta)$ are then identical and the action* (3.4.2) *endows $\mathcal{T}(\theta)$ with the structure of principal homogeneous space over $X_1(T)$.*

Proof. The character $\phi^{\boldsymbol{J}}$ in (3.4.1) is trivial on J^1_θ while its restriction to $G_P \cap \boldsymbol{J}_\theta$ is of the form $\chi \circ \det_P$, for some $\chi \in X_1(P)$ (as in the proof of 3.1 Proposition). Part (1) then follows from the definition of maximal simple type and 3.2 Remark. In part (2), the first assertion follows from the definition of maximal simple type and the second is a case of 3.2 Corollary. $\square$

3.5. Consider the case where the m-simple character θ is *trivial.* That is, θ is the trivial character of $U^1_{\mathfrak{m}}$, where $\mathfrak{m}$ is a *maximal* $\mathfrak{o}_F$-order in $A = \mathrm{M}_n(F)$. By

definition [17] (5.5.10), the set $\mathcal{T}(\theta)$ consists of equivalence classes of representations Λ of $\boldsymbol{J}_\theta = F^\times U_{\mathfrak{m}}$, trivial on $U^1_{\mathfrak{m}}$, such that $\Lambda|_{U_{\mathfrak{m}}}$ is the inflation of an irreducible *cuspidal* representation of $U_{\mathfrak{m}}/U^1_{\mathfrak{m}} \cong \mathrm{GL}_n(\Bbbk_F)$.

The set $\mathcal{T}(\theta)$ carries a natural action of the group $X_1(F)$, denoted $(\chi, \Lambda) \mapsto \chi\Lambda$, where $\chi\Lambda : x \mapsto \chi(\det x)\Lambda(x)$, $x \in \boldsymbol{J}_\theta$. Since θ has parameter field F, this action is the same as that given by (3.4.2).

Any two choices of θ (that is, of $\mathfrak{m}$) are G-conjugate. Conjugation by an element g of G induces an $X_1(F)$-bijection $\mathcal{T}(\theta) \to \mathcal{T}(\theta^g)$, so the $X_1(F)$-set $\mathcal{T}(\theta)$ essentially depends only on the dimension n. We shall therefore write $\mathcal{T}(\theta) = \mathcal{T}_n(\mathbf{0}_F)$ in this case.

To describe the elements of $\mathcal{T}_n(\mathbf{0}_F)$ concretely, it is therefore enough to treat the standard example $\mathfrak{m} = \mathrm{M}_n(\mathfrak{o}_F)$. We take an unramified field extension E/F of degree n, and set $\Delta = \mathrm{Gal}(E/F)$. The group Δ acts on $X_1(E)$. We denote by $X_1(E)^{\Delta\text{-reg}}$ the set of Δ-regular elements of $X_1(E)$ and by $X_1(E)^{\Delta}$ the set of Δ-fixed points. We identify E with a subfield of A, via some F-embedding, such that $E^\times \subset \boldsymbol{J}_\theta = F^\times U_{\mathfrak{m}}$. Any two such embeddings are conjugate under $U_{\mathfrak{m}}$, so the choice will be irrelevant.

Proposition. *Let $\phi \in X_1(E)^{\Delta\text{-reg}}$.*

(1) *There exists a unique representation $\lambda_\phi \in \mathcal{T}_n(\mathbf{0}_F)$ such that*

$$\operatorname{tr}\lambda_\phi(z\zeta) = (-1)^{n-1}\phi(z)\sum_{\delta\in\Delta}\phi^\delta(\zeta), \tag{3.5.1}$$

for $z \in F^\times$ and every Δ-regular element ζ of $\boldsymbol{\mu}_E$.

(2) *The representation λ_ϕ depends, up to equivalence, only on the Δ-orbit of ϕ and the map $\phi \mapsto \lambda_\phi$ induces a canonical bijection*

$$\Delta\backslash X_1(E)^{\Delta\text{-reg}} \xrightarrow{\approx} \mathcal{T}_n(\mathbf{0}_F). \tag{3.5.2}$$

(3) *If $\chi \in X_1(F)$, write $\chi_E = \chi\circ \mathrm{N}_{E/F} \in X_1(E)$. The map $\chi \mapsto \chi_E$ is an isomorphism $X_1(F)/X_0(F)_n \to X_1(E)^{\Delta}$ and, if $\phi \in X_1(E)^{\Delta\text{-reg}}$, then*

$$\lambda_{\chi_E\phi} \cong \chi\lambda_\phi, \quad \chi \in X_1(F). \tag{3.5.3}$$

Proof. Parts (1) and (2) re-state, for example, [10] 2.2 Proposition. Part (3) is elementary. $\square$

For a more detailed examination of this classical construction and further references, see [13] §2.

3.6. We return to a *non-trivial* m-simple character θ in $G = \mathrm{GL}_n(F)$ with parameter field P/F of degree $n/s = \deg cl(\theta)$. We let T/F be the tame parameter field inside P. We re-interpret the definition of extended maximal simple type by defining a pairing

$$\mathcal{T}_s(\mathbf{0}_T) \times \mathcal{H}(\theta) \longrightarrow \mathcal{T}(\theta). \tag{3.6.1}$$

To do this, we take $\lambda \in \mathcal{T}_s(\mathbf{0}_T)$. Let P_s/P be unramified of degree s and choose a P-embedding of P_s in A such that $P_s^\times \subset \boldsymbol{J}_\theta$. Let T_s/F be the maximal tamely ramified sub-extension of P_s/F. Thus T_s/T is unramified of degree s, and we may identify $\varDelta = \mathrm{Gal}(P_s/P)$ with $\mathrm{Gal}(T_s/T)$. There is then a character $\xi \in X_1(T)^{\varDelta\text{-reg}}$ such that $\lambda \cong \lambda_\xi$.

We next set $\xi_P = \xi \circ \mathrm{N}_{P_s/T_s} \in X_1(P_s)^{\varDelta\text{-reg}}$. We so obtain a representation $\lambda_{\xi_P} \in \mathcal{T}_s(\mathbf{0}_P)$. Since $\mathfrak{a}_P \cong \mathrm{M}_s(\mathfrak{o}_P)$, we may think of λ_{ξ_P} as a representation of $P^\times U_{\mathfrak{a}_P}$ trivial on $U^1_{\mathfrak{a}_P}$. As $\boldsymbol{J}_\theta = P^\times U_{\mathfrak{a}_P} J^1_\theta$ and $P^\times U_{\mathfrak{a}_P} \cap J^1_\theta = U^1_{\mathfrak{a}_P}$, we may extend λ_{ξ_P} to a representation of $\boldsymbol{J}_\theta$ trivial on J^1_θ. We denote this representation $\lambda_\xi^{\boldsymbol{J}}$.

We may specify the representation $\lambda_\xi^{\boldsymbol{J}}$ without reference to the parameter field P/F as follows. We observe that $P^\times J^1_\theta$ is the inverse image, in $\boldsymbol{J}_\theta$, of the centre of the group $\boldsymbol{J}_\theta/J^1_\theta$.

Lemma. *Let T_s/T be unramified of degree s, such that $T_s^\times \subset \boldsymbol{J}_\theta$. Let $\varDelta = \mathrm{Gal}(T_s/T)$. The representation $\lambda_\xi^{\boldsymbol{J}}$ has the following properties:*

(a) *$\lambda_\xi^{\boldsymbol{J}}$ is trivial on J^1_θ;*

(b) *the restriction of $\lambda_\xi^{\boldsymbol{J}}$ to $P^\times J^1_\theta$ is a multiple of the character $(\xi|_{T^\times})^{\boldsymbol{J}} \in X_1(\theta)$ (cf.* (3.1.1)*);*

(c) *if $\zeta \in \boldsymbol{\mu}_{T_s}$ is $\varDelta$-regular, then*

$$\mathrm{tr}\,\lambda_\xi^{\boldsymbol{J}}(\zeta) = (-1)^{s-1} \sum_{\delta\in\varDelta} \xi^\delta(\zeta^{p^r}),$$

where $p^r = [P{:}T] = \deg cl(\theta)/[T{:}F]$.

We now form the representation

$$\lambda_\xi \ltimes \kappa = \lambda_\xi^{\boldsymbol{J}} \otimes \kappa \tag{3.6.2}$$

of $\boldsymbol{J}_\theta$. Re-assembling all the definitions and recalling 3.2 Proposition, we find:

Proposition. *Let $\kappa \in \mathcal{H}(\theta)$, $\lambda \in \mathcal{T}_s(\mathbf{0}_T)$. The representation $\lambda \ltimes \kappa$ of $\boldsymbol{J}_\theta$ lies in $\mathcal{T}(\theta)$.*

(1) *If $\phi \in X_1(T)$, $\kappa \in \mathcal{H}(\theta)$ and $\lambda \in \mathcal{T}_s(\mathbf{0}_T)$ then*

$$\phi \odot (\lambda \ltimes \kappa) \cong (\phi\,\lambda) \ltimes \kappa \cong \lambda \ltimes (\phi \odot \kappa).$$

(2) *For any $\kappa \in \mathcal{H}(\theta)$, the map*

$$\begin{aligned} \mathcal{T}_s(\mathbf{0}_T) &\longrightarrow \mathcal{T}(\theta), \\ \lambda &\longmapsto \lambda \ltimes \kappa, \end{aligned}$$

is a bijection.

Proof. The first assertion, and the surjectivity of the map in part (2), are the definition of maximal simple type. Part (1) is also immediate from the definitions, while part (2) follows from part (1) and 3.2 Proposition. □

Remark. Observe the similarity between part (2) of the proposition and 1.6 Lemma.

4. Cuspidal representations

Let θ be an m-simple character in $G = \mathrm{GL}_n(F)$, with tame parameter field T/F. In §3, we described a canonical action of $X_1(T)$ on the set $\mathcal{T}(\theta)$ of equivalence classes of extended maximal simple types in G defined by θ. We wish to translate this into a structure on the set of equivalence classes of irreducible cuspidal representations of G which contain θ. To do this, we must work with the endo-class $\varTheta = cl(\theta)$ and a tame parameter field E/F for $\varTheta$, taking appropriate care with identifications.

4.1. If $n \geqslant 1$ is an integer, let $\mathcal{A}_n^0(F)$ denote the set of equivalence classes of irreducible *cuspidal* representations of $\mathrm{GL}_n(F)$. We summarize the basic results, from 6.2 and 8.4 of [17], concerning the classification of the elements of $\mathcal{A}_n^0(F)$ via simple types.

A representation $\pi \in \mathcal{A}_n^0(F)$ contains an m-simple character θ in $G = \mathrm{GL}_n(F)$ and, up to G-conjugacy, only one. The endo-class $cl(\theta)$ therefore depends only on the equivalence class of π. We accordingly write

$$\vartheta(\pi) = cl(\theta). \tag{4.1.1}$$

If $\Theta \in \mathcal{E}(F)$ and if $m \geqslant 1$ is an integer, we put $n = m \deg \Theta$ and define

$$\mathcal{A}_m^0(F;\Theta) = \{\pi \in \mathcal{A}_n^0(F) : \vartheta(\pi) = \Theta\}. \tag{4.1.2}$$

If we choose an m-realization θ of Θ in $G = \mathrm{GL}_n(F)$, a representation $\pi \in \mathcal{A}_n^0(F)$ lies in $\mathcal{A}_m^0(F;\Theta)$ if and only if π contains θ (*cf.* 2.5 Lemma 2). If $\Lambda \in \mathcal{T}(\theta)$, the representation $\pi_\Lambda = c\text{-}\mathrm{Ind}_{\boldsymbol{J}_\theta}^G \Lambda$ is irreducible, cuspidal and contains θ: that is, $\pi_\Lambda \in \mathcal{A}_m^0(F;\Theta)$. The map

$$\begin{aligned} \mathcal{T}(\theta) &\longrightarrow \mathcal{A}_m^0(F;\Theta), \\ \Lambda &\longmapsto c\text{-}\mathrm{Ind}_{\boldsymbol{J}_\theta}^G \Lambda, \end{aligned} \tag{4.1.3}$$

is a bijection.

Remark. Let $\pi \in \mathcal{A}_n^0(F)$ contain a simple character φ. One may show that φ is m-simple and $cl(\varphi) = \vartheta(\pi)$. We have no use here for this fact, so we give no proof.

4.2. Let $\Theta \in \mathcal{E}(F)$ have degree d, let E/F be a tame parameter field for Θ, and let Ψ be a totally wild E/F-lift of Θ.

Let $m \geqslant 1$ be an integer, and set $n = md$. We take a pair $(\theta, \boldsymbol{\iota})$ consisting of an m-realization θ of Θ in $G = \mathrm{GL}_n(F)$ and an F-embedding $\boldsymbol{\iota}$ of E in $\mathrm{M}_n(F)$ such that $\boldsymbol{\iota}E/F$ is a tame parameter field for θ satisfying $\boldsymbol{\iota}_*\Psi = cl(\theta_{\boldsymbol{\iota}E})$. Such a pair $(\theta, \boldsymbol{\iota})$ exists and is uniquely determined, up to conjugation in G by an element commuting with $\boldsymbol{\iota}E$ (2.7 Proposition). We first use $\boldsymbol{\iota}$ to translate the natural action (3.4.2) of $X_1(\boldsymbol{\iota}E)$ on $\mathcal{T}(\theta)$ into an action of $X_1(E)$. The induction

relation (4.1.3) then turns this into an action of $X_1(E)$ on $\mathcal{A}^0_m(F;\Theta)$. In all, we get a *canonical* pairing

$$\begin{aligned} X_1(E) \times \mathcal{A}^0_m(F;\Theta) &\longrightarrow \mathcal{A}^0_m(F;\Theta), \\ (\phi,\pi) &\longmapsto \phi \odot_\Psi \pi, \end{aligned} \tag{4.2.1}$$

satisfying $(\phi\phi') \odot_\Psi \pi = \phi \odot_\Psi (\phi' \odot_\Psi \pi)$. This action of $X_1(E)$ does depend on Ψ, but on no other choices.

In the case $E = F$, we have $\Theta = \Psi$ and the $\odot_\Psi$-action of $X_1(F)$ on $\mathcal{A}^0_m(F;\Theta)$ is the standard one $(\phi,\pi) \mapsto \phi\pi$, where $\phi\pi : x \mapsto \phi(\det x)\,\pi(x)$.

Another special case is worthy of note. As a direct consequence of 3.4 Proposition, we have:

Proposition. *The action $\odot_\Psi$ endows $\mathcal{A}^0_1(F;\Theta)$ with the structure of principal homogeneous space over $X_1(E)$.*

Remark. In the general situation, let Ψ' be some other totally wild E/F-lift of Θ. Thus there exists $\gamma \in \mathrm{Aut}(E|F)$ such that $\Psi' = \Psi^\gamma$ (2.4 Corollary). We then get the relation

$$\phi^\gamma \odot_{\Psi^\gamma} \pi = \phi \odot_\Psi \pi, \quad \phi \in X_1(E),\ \pi \in \mathcal{A}^0_m(F;\Theta). \tag{4.2.2}$$

Indeed, this relation holds for any F-isomorphism $\gamma : E \to E^\gamma$. (Observe the apparent parallel with the discussion in 1.5, especially (1.5.3).)

5. Algebraic induction maps

In §1, we saw that the irreducible representations of the Weil group $\mathcal{W}_F$ can be constructed in a uniform manner, using only very particular induction processes (as in 1.6 Proposition). In this section, we define analogous "algebraic induction maps" on the other side.

5.1. To start, we recall something of the Glauberman correspondence [20], as developed in Appendix 1 to [4]. In this sub-section, we use a system of notation distinct from that of the rest of the paper.

Let G be a finite group, and let $\mathrm{Irr}\,G$ denote the set of equivalence classes of irreducible representations of G. Let A be a finite soluble group and $A \to \mathrm{Aut}\,G$

a homomorphism with image of order relatively prime to that of G. Let G^A denote the group of A-fixed points in G. The group A acts on the set $\operatorname{Irr} G$, so let $\operatorname{Irr}^A G$ denote the set of fixed points. The Glauberman correspondence is a canonical bijection

$$\mathsf{g}_G^A : \operatorname{Irr}^A G \longrightarrow \operatorname{Irr} G^A,$$

depending only on the image of A in $\operatorname{Aut} G$, which is transitive in the following sense. Let B be a normal subgroup of A. Thus A/B acts on G^B, and similarly on representations. We then have

$$\mathsf{g}_G^A = \mathsf{g}_{G^B}^{A/B} \circ \mathsf{g}_G^B. \tag{5.1.1}$$

To describe the correspondence g_G^A, it is therefore enough to consider the case where A is *cyclic.*

(5.1.2) *Suppose A is cyclic, of order relatively prime to $|G|$. Let $\rho \in \operatorname{Irr}^A G$.*

(1) *There is a unique representation $\tilde\rho$ of $A \ltimes G$ such that $\tilde\rho|_G \cong \rho$ and* $\det \tilde\rho|_A = 1$.

(2) *There is a unique $\rho^A \in \operatorname{Irr} G^A$ such that*

$$\operatorname{tr} \rho^A(h) = \epsilon \operatorname{tr} \tilde\rho(ah), \tag{5.1.3}$$

for all $h \in G^A$, any generator a of A, and a constant ϵ.

(3) *The constant ϵ has the value ± 1.*

In this context, the representation ρ^A is $\mathsf{g}_G^A(\rho)$.

The constant ϵ depends on both ρ and the image of A in $\operatorname{Aut} G$: different cyclic operator groups on G may have the same fixed points and give the same character correspondence, but for different constants ϵ. An instance of this occurs in 5.4 below.

5.2. We return to our standard situation and notation.

A *complementary subgroup* of $F^\times$ is a closed subgroup C of $F^\times$ such that the product map $C \times U_F^1 \to F^\times$ is bijective. A complementary subgroup C is necessarily of the form $C = C_F(\varpi_F) = \langle \varpi_F, \boldsymbol{\mu}_F \rangle$, for some prime element ϖ_F of F. If ϖ_F, ϖ_F' are prime elements of F, then $C_F(\varpi_F') = C_F(\varpi_F)$ if and only if $\varpi_F' = \alpha\varpi_F$, for some $\alpha \in \boldsymbol{\mu}_F$.

Lemma. *Let E/F be a finite, tamely ramified field extension, and let ϖ_F be a prime element of F.*

(1) *There exists a unique complementary subgroup $C_E(\varpi_F)$ of E such that $\varpi_F \in C_E(\varpi_F)$.*
(2) *Let $F \subset L \subset E$. The group $C_E(\varpi_F) \cap L^\times$ is the complementary subgroup $C_L(\varpi_F)$ of $L^\times$ containing ϖ_F.*

Proof. Let ϖ be a prime element of E, and set $e = e(E|F)$. Thus $\varpi^e = \zeta\varpi_F u$, for some $\zeta \in \boldsymbol{\mu}_E$ and $u \in U_E^1$. Since p does not divide e, there exists a unique element v of U_E^1 such that $v^e = u$. Replacing ϖ by $v^{-1}\varpi$, we may assume $\varpi^e \in \boldsymbol{\mu}_E\varpi_F$. The group $\langle\varpi, \boldsymbol{\mu}_E\rangle$ is a complementary subgroup of $E^\times$ containing ϖ_F. It is clearly the unique such subgroup with this property. Assertion (2) is now immediate. □

5.3. We define a family of "algebraic induction maps". The construction proceeds in two steps, reflecting the discussion in 5.1. For the first, we work in the following special situation.

Notation.

(1) *θ is a simple character in $G = \mathrm{GL}_n(F)$,*
(2) *$\theta \in \mathcal{C}(\mathfrak{a}, \beta)$, for a simple stratum $[\mathfrak{a}, \beta]$ in $A = \mathrm{M}_n(F)$ such that $P = F[\beta]/F$ is* totally ramified of degree *n;*
(3) *E/F is the maximal tamely ramified sub-extension of P/F;*
(4) *$[E{:}F] = e$ and $[P{:}E] = p^r$.*

The simple character θ is necessarily m-simple. Consequently, the set $\mathcal{T}(\theta) = \mathcal{H}(\theta)$ is a principal homogeneous space over $X_1(E)$ (3.4 Proposition).

Let η be the 1-Heisenberg representation of $J^1 = J^1(\beta, \mathfrak{a})$ over θ. The natural conjugation action of $E^\times$ on J^1 induces an action of the group $M_{E/F} = E^\times/F^\times U_E^1$ on the finite p-group $J^1/\mathrm{Ker}\,\theta$. The group $M_{E/F}$ is cyclic, of order $e = [E{:}F]$ (which is relatively prime to p). The group of $M_{E/F}$-fixed points in $J^1/\mathrm{Ker}\,\theta$ is $J_E^1/\mathrm{Ker}\,\theta_E$ (*cf.* 4.1 Lemma 1 of [4]).

Taking complementary subgroups $C_F(\varpi_F) \subset C_E(\varpi_F)$ as in 5.2, the canonical map $C_E(\varpi_F) \to M_{E/F}$ induces an isomorphism $C_E(\varpi_F)/C_F(\varpi_F) \cong M_{E/F}$. We

may therefore switch, as convenient, between $C_E(\varpi_F)$ and $M_{E/F}$ as operator groups.

As in 2.3, G_E denotes the G-centralizer of $E^\times$. Using the standard notation of that paragraph, let η_E denote the 1-Heisenberg representation of $J^1_E = J^1 \cap G_E = J^1(\beta, \mathfrak{a}_E)$ over $\theta_E = \theta|_{H^1_E}$. For the next result, we initially view η as a representation of the finite p-group $J^1/\mathrm{Ker}\,\theta$, and similarly for η_E.

Lemma.

(1) *There exists a unique representation* $\tilde\eta$ *of* $M_{E/F} \ltimes J^1/\mathrm{Ker}\,\theta$ *extending* η *and such that* $\det\tilde\eta|_{M_{E/F}} = 1$.

(2) *There is a constant* $\epsilon = \epsilon_{E/F} = \pm 1$ *such that*

$$\mathrm{tr}\,\tilde\eta(zj) = \epsilon\,\mathrm{tr}\,\eta_E(j), \tag{5.3.1}$$

for every $j \in J^1_E$ *and every generator* z *of the cyclic group* $M_{E/F}$.

(3) *The sign* ϵ *depends only on the simple stratum* $[\mathfrak{a}, \beta]$ *(or on* θ*), and not on the choice of tame parameter field* E/F.

Proof. The first assertion is elementary (*cf.* (5.1.2)). In (2), the Glauberman correspondence gives a unique irreducible representation ζ of $J^1_E/\mathrm{Ker}\,\theta_E$ such that $\mathrm{tr}\,\zeta(j) = \epsilon\,\mathrm{tr}\,\tilde\eta(zj)$, for z and j as above. We inflate $\tilde\eta$, ζ to representations of $C_E(\varpi_F) \ltimes J^1$, J^1_E respectively. Replacing j by jh, $h \in H^1_E$, we find that $\mathrm{tr}\,\zeta(jh) = \theta_E(h)\,\mathrm{tr}\,\zeta(j)$. Thus ζ contains θ_E, whence $\zeta \cong \eta_E$, as required.

The sign $\epsilon = \epsilon_{E/F}$ can be computed using [2] 8.6.1. We form the finite, elementary abelian p-group $V = V_\theta = J^1/H^1$. This carries a nondegenerate alternating form over $\mathbb{F}_p$, as in [17] 5.1. The conjugation action of $E^\times$ on J^1 induces an action of $M_{E/F}$ on V, and V provides a symplectic representation of $M_{E/F}$ over $\mathbb{F}_p$. In the notation of [13] §3, $\epsilon_{E/F}$ is given by

$$\epsilon_{E/F} = t_{M_{E/F}}(V). \tag{5.3.2}$$

That is, $\epsilon_{E/F}$ is an invariant of the $M_{E/F}$-module V (*cf.* [13] 3.3), and V is determined directly from $[\mathfrak{a}, \beta]$ (or from θ). Any two choices of E are J^1-conjugate (2.6 Proposition), and such a conjugation has no effect on the module structure of V. $\square$

Proposition. *Define* $\epsilon = \epsilon_{E/F} = \pm 1$ *by* (5.3.1). *Let* $\Lambda \in \mathcal{T}(\theta)$. *There exists a unique* $\Lambda_E \in \mathcal{T}(\theta_E)$ *such that*

$$\operatorname{tr} \Lambda_E(x) = \epsilon_{E/F} \operatorname{tr} \Lambda(x), \tag{5.3.3}$$

for all $x \in \boldsymbol{J}_E$ *such that* $\upsilon_E(\det_E x) = \upsilon_F(\det x)$ *is relatively prime to* $[E{:}F]$. *The map*

$$\begin{aligned} \mathcal{T}(\theta) &\longrightarrow \mathcal{T}(\theta_E), \\ \Lambda &\longmapsto \Lambda_E, \end{aligned} \tag{5.3.4}$$

is a bijection, and an isomorphism of $X_1(E)$*-spaces.*

Proof. If $\Lambda \in \mathcal{T}(\theta)$, a representation Λ_E of $\boldsymbol{J}_E$ satisfying (5.3.3) must lie in $\mathcal{T}(\theta_E)$, and is uniquely determined by Λ. Moreover, if we have such a pair and if $\phi \in X_1(E)$, then (in the notation of 3.2)

$$\operatorname{tr} \phi \odot \Lambda_E(x) = \epsilon_{E/F} \operatorname{tr} \phi \odot \Lambda(x),$$

for all x as before. In this situation, each of the sets $\mathcal{T}(\theta)$, $\mathcal{T}(\theta_E)$ is a principal homogeneous space over $X_1(E)$. The result will follow, therefore, if we produce one representation $\Lambda \in \mathcal{T}(\theta)$ for which there exists Λ_E satisfying (5.3.3).

To do this, we choose a prime element ϖ_F of F and let $C_E = C_E(\varpi_F)$ be the unique complementary subgroup of $E^\times$ containing ϖ_F. Let ${}_p\boldsymbol{J} = {}_p\boldsymbol{J}(\varpi_F)$ denote the inverse image in $\boldsymbol{J}$ of the unique Sylow pro-p subgroup of the profinite group $\boldsymbol{J}/\langle\varpi_F\rangle$. Thus $\boldsymbol{J} = C_E \cdot {}_p\boldsymbol{J}$. Also, ${}_p\boldsymbol{J}_E = {}_p\boldsymbol{J} \cap G_E$ is the inverse image in $\boldsymbol{J}_E$ of the Sylow pro-p subgroup of $\boldsymbol{J}_E/\langle\varpi_F\rangle$.

We choose $\Lambda \in \mathcal{T}(\theta)$ such that $\Lambda(\varpi_F) = 1$ and $\det \Lambda$ is trivial on C_E. Since $\dim \Lambda$ is a power of p and $\boldsymbol{\mu}_F$ is central, these conditions imply that Λ is trivial on $\boldsymbol{\mu}_F$, hence also on $C_F = C_E \cap F^\times$. We may therefore view Λ as the inflation of a representation of $C_E/C_F \ltimes {}_p\boldsymbol{J}/\langle\varpi_F\rangle$. The Glauberman correspondence gives a representation $\tilde{\Lambda}_E$ of ${}_p\boldsymbol{J}_E/\langle\varpi_F\rangle$ such that

$$\operatorname{tr} \tilde{\Lambda}_E(x) = \epsilon \operatorname{tr} \Lambda(cx), \quad x \in {}_p\boldsymbol{J}_E/\langle\varpi_F\rangle,$$

for any generator c of the cyclic group C_E/C_F. Moreover, $\tilde{\Lambda}_E|_{J^1_E} \cong \eta_E$. We extend $\tilde{\Lambda}_E$, by triviality, to a representation of $C_E/C_F \times {}_p\boldsymbol{J}_E/\langle\varpi_F\rangle$ and then inflate it to a representation Λ_E of $C_E \cdot {}_p\boldsymbol{J}_E = \boldsymbol{J}_E$. This representation satisfies (5.3.3). $\square$

We interpret the proposition in terms of cuspidal representations of linear groups, following the discussion in §4.

Corollary 1. *Let $\Theta = cl(\theta)$, $\Theta_E = cl(\theta_E)$. The map* (5.3.4) *induces a canonical bijection*

$$\begin{aligned} \mathcal{A}_1^0(F;\Theta) &\longrightarrow \mathcal{A}_1^0(E;\Theta_E), \\ \pi &\longmapsto \pi_E, \end{aligned} \tag{5.3.5}$$

such that

$$(\phi \odot_{\Theta_E} \pi)_E = \phi\pi_E, \tag{5.3.6}$$

for $\phi \in X_1(E)$ and $\pi \in \mathcal{A}_1^0(F;\Theta)$.

Proof. We start from the endo-classes $\Theta_E \in \mathcal{E}(E)$ and $\Theta \in \mathcal{E}(F)$. Thus Θ_E is totally wild, $\Theta = \mathfrak{i}_{E/F}\Theta_E$ (*cf.* (2.3.4)), and E/F is a tame parameter field for Θ.

We identify E with a subfield of $\mathrm{M}_n(F)$. Following 2.7 Proposition, we choose a realization θ of Θ in $G = \mathrm{GL}_n(F)$ such that E/F is a tame parameter field for θ and $cl(\theta_E) = \Theta_E$. Thus θ is uniquely determined up to conjugation by an element of G_E.

By definition, π contains a representation $(\boldsymbol{J}_\theta, \Lambda) \in \mathcal{T}(\theta)$, giving $c\text{-}\mathrm{Ind}_{\boldsymbol{J}_\theta}^G \Lambda \cong \pi$. Using the map (5.3.4), we form the representation $\Lambda_E \in \mathcal{T}(\theta_E)$ and set $\pi_E = c\text{-}\mathrm{Ind}_{\boldsymbol{J}_{\theta_E}}^{G_E} \Lambda_E$. The G_E-conjugacy class of Λ_E is then independent of the choice of θ, so $\pi \mapsto \pi_E$ gives a canonical map $\mathcal{A}_1^0(F;\Theta) \to \mathcal{A}_1^0(E;\Theta_E)$. The bijectivity of this map and property (5.3.6) then follow from the proposition. □

We let

$$ind_{E/F} : \mathcal{A}_1^0(E;\Theta_E) \longrightarrow \mathcal{A}_1^0(F;\Theta) \tag{5.3.7}$$

denote the inverse of the map (5.3.5). Thus $ind_{E/F}$ is a bijection satisfying

$$ind_{E/F}\, \phi\rho = \phi \odot_{\Theta_E} ind_{E/F}\, \rho, \quad \rho \in \mathcal{A}_1^0(E;\Theta_E),\ \phi \in X_1(E). \tag{5.3.8}$$

It also has an important naturality property.

Corollary 2. *If $\gamma : E \to E^\gamma$ is an isomorphism of local fields, then*

$$ind_{E^\gamma/F^\gamma}\, \rho^\gamma = \big(ind_{E/F}\, \rho\big)^\gamma,$$

for all $\rho \in \mathcal{A}_1^0(E;\Theta_E)$.

Proof. To construct the representation $ind_{E/F}\,\rho$ more directly, we first choose an F-embedding of E in $\mathrm{M}_n(F)$. We then choose a realization φ of Θ_E in G_E, and form the simple character φ^F in G, as in (2.3.3). Let $\lambda \in \mathcal{T}(\varphi)$ occur in ρ. There is a unique $\lambda^F \in \mathcal{T}(\varphi^F)$ such that $(\lambda^F)_E \cong \lambda$. The representation $ind_{E/F}\,\rho$ is then equivalent to c-Ind λ^F. Changing the embedding of E in $\mathrm{M}_n(F)$ or the realization φ only replaces λ^F by a G-conjugate. It follows that $\big(ind_{E^\gamma/F^\gamma}\,\rho^\gamma\big)^{\gamma^{-1}} \cong ind_{E/F}\,\rho$, as required. $\square$

5.4. For the second step in the construction, we pass to the general situation. The notation that follows will be standard for the rest of the section.

Notation.

(1) $[\mathfrak{a},\beta]$ *is an* m*-simple stratum in* $A = \mathrm{M}_n(F)$ *and* $\theta \in \mathcal{C}(\mathfrak{a},\beta,\psi_F)$*;*
(2) $P_0 = F[\beta]$ *and* $m = n/[P_0{:}F]$*;*
(3) E_0/F *is the maximal tamely ramified sub-extension of* P_0/F *and* K_0/F *is its maximal unramified sub-extension;*
(4) $n_0 = [E_0{:}F]$, $e = e(E_0|F)$ *and* $p^r = [P_0{:}E_0]$*;*
(5) P/P_0 *is unramified of degree* m *and* $P^\times \subset \boldsymbol{J}(\beta,\mathfrak{a}) = \boldsymbol{J}_\theta$*;*
(6) K/F *is the maximal unramified sub-extension of* P/F *and* $E = E_0K/F$ *its maximal tamely ramified sub-extension;*
(7) $\Gamma = \mathrm{Gal}(K/F)$ *and* $\Delta = \mathrm{Gal}(P/P_0) \cong \mathrm{Gal}(E/E_0) \cong \mathrm{Gal}(K/K_0)$.

As usual, we put $G = \mathrm{GL}_n(F)$. Any two extensions P/P_0, satisfying the conditions (5), are J_θ^0-conjugate: as we will see (Comments, 5.7), this means that the choice of P/P_0 is irrelevant. We use the notational scheme of 2.3. In particular, G_K is the G-centralizer of $K^\times$ and $\theta_K \in \mathcal{C}(\mathfrak{a}_K,\beta,\psi_K)$. The extension E/K is thus a tame parameter field for θ_K.

Our immediate aim is to define a canonical $X_1(E_0)$-bijection

$$\ell_{K/F} : \mathcal{H}(\theta) \xrightarrow{\approx} \mathcal{H}(\theta_K)^\Delta.$$

This will be achieved in 5.6 Proposition.

We apply the Glauberman correspondence to the action of $\boldsymbol{\mu}_K$ on $J^1 = J^1(\beta,\mathfrak{a})$.

Lemma. *Let $\tilde\eta$ be the representation of $\boldsymbol{\mu}_K \ltimes J^1$ such that $\tilde\eta|_{J^1} \cong \eta$ and* $\det\tilde\eta|_{\boldsymbol{\mu}_K} = 1$. *There is a constant $\epsilon^0_{\boldsymbol{\mu}_K} = \pm 1$ and a character $\epsilon^1_{\boldsymbol{\mu}_K} : \boldsymbol{\mu}_K \to \{\pm 1\}$ such that*

$$\operatorname{tr}\eta_K(x) = \epsilon^0_{\boldsymbol{\mu}_K}\epsilon^1_{\boldsymbol{\mu}_K}(\zeta)\operatorname{tr}\tilde\eta(\zeta x), \tag{5.4.1}$$

for $x \in J^1_K$ and any Γ-regular element ζ of $\boldsymbol{\mu}_K$. The quantities $\epsilon^j_{\boldsymbol{\mu}_K}$ depend only on θ (or on the simple stratum $[\mathfrak{a},\beta]$).

Proof. Just as in 5.3 Lemma, there is a constant $\epsilon(\boldsymbol{\mu}_K) = \pm 1$ such that

$$\operatorname{tr}\eta_K(x) = \epsilon(\boldsymbol{\mu}_K)\operatorname{tr}\tilde\eta(\zeta x),$$

for $x \in J^1_K$ and any *generator* ζ of $\boldsymbol{\mu}_K$. More generally, let $\zeta \in \boldsymbol{\mu}_K$ be Γ-regular. The set of ζ-fixed points in J^1 is then J^1_K and

$$\operatorname{tr}\eta_K(x) = \epsilon(\langle\zeta\rangle)\operatorname{tr}\tilde\eta(\zeta x), \quad x \in J^1_K,$$

where $\epsilon(\langle\zeta\rangle) = \pm 1$ depends on the subgroup of $\boldsymbol{\mu}_K$ generated by ζ. According to [2] 8.6.1 (but using the notation of [13] §3), we have $\epsilon(\langle\zeta\rangle) = t_{\langle\zeta\rangle}(V)$, where $V = J^1/H^1$. However, [13] Proposition 3.6 gives

$$\epsilon(\langle\zeta\rangle) = \epsilon^0_{\boldsymbol{\mu}_K}\,\epsilon^1_{\boldsymbol{\mu}_K}(\zeta),$$

where

$$\epsilon^j_{\boldsymbol{\mu}_K} = t^j_{\boldsymbol{\mu}_K}(V), \quad j = 0, 1. \tag{5.4.2}$$

In particular, these quantities depend only on the simple stratum $[\mathfrak{a},\beta]$. $\square$

Remark. The character $\epsilon^1_{\boldsymbol{\mu}_K}$ is, by its definition in terms of the conjugation action of $\boldsymbol{\mu}_K$ on V, trivial on $\boldsymbol{\mu}_F$. Moreover, the definition of $t^1_{\boldsymbol{\mu}_K}(V)$ in [13] 3.4 Definition 3 shows that $\epsilon^1_{\boldsymbol{\mu}_K}$ is trivial when $p = 2$.

5.5. We choose a prime element ϖ_F of F. As in 5.2 Lemma, let $C_{E_0}(\varpi_F)$ (resp. $C_E(\varpi_F)$) be the unique complementary subgroup of $E_0^\times$ (resp. $E^\times$) containing ϖ_F.

We consider the group $P_0^\times J^1$, described more intrinsically as the inverse image in $\boldsymbol{J}$ of the centre of the group $\boldsymbol{J}/J^1$. The group $P_0^\times J^1/\langle\varpi_F\rangle$ is profinite, and

has a unique Sylow pro-p subgroup. We denote by ${}_p\boldsymbol{J}$ the inverse image of this subgroup in $\boldsymbol{J}$. *The definition of ${}_p\boldsymbol{J}$ depends on the choice of ϖ_F:* we accordingly use the notation ${}_p\boldsymbol{J} = {}_p\boldsymbol{J}(\varpi_F)$ when it is necessary to emphasize this fact. The quotient ${}_p\boldsymbol{J}(\varpi_F)/J^1$ is cyclic, generated by an element ϖ_0 of P_0 such that $\varpi_0^{p^r} \equiv \varpi_F \pmod{U^1_{P_0}}$.

Using the quotient $P^\times J^1_K/\langle\varpi_F\rangle$, we may similarly define a subgroup ${}_p\boldsymbol{J}_K = {}_p\boldsymbol{J}_K(\varpi_F)$ of $\boldsymbol{J}_K$. This satisfies ${}_p\boldsymbol{J}_K = {}_p\boldsymbol{J} \cap \boldsymbol{J}_K$.

We will need to apply the Conjugacy Lemma of 2.6 in this new context. We must therefore verify:

Lemma. *The pro-p group ${}_p\boldsymbol{J}/\langle\varpi_F\rangle$ admits a $P^\times$-stable filtration satisfying the conditions of* (2.6).

Proof. We have to refine the standard filtration of the group J^1. A generator, ω say, of ${}_p\boldsymbol{J}/J^1$ acts on each step J^k/J^{k+1} of the standard filtration as a unipotent automorphism commuting with the natural action of $P^\times$. We therefore insert into the standard filtration the extra steps $\operatorname{Ker}(\omega-1)^r|_{J^k/J^{k+1}}$. $\square$

5.6. We start by noting a consequence of 3.2 Proposition.

Lemma 1. *Let $\kappa_1, \kappa_2 \in \mathcal{H}(\theta)$. If $\kappa_1|_{P^\times J^1} \cong \kappa_2|_{P^\times J^1}$, then $\kappa_1 \cong \kappa_2$.*

We apply the Glauberman correspondence to representations of ${}_p\boldsymbol{J}$. We first identify a family of special elements of $\mathcal{H}(\theta)$.

Lemma 2. *Let ϖ_F be a prime element of F. There exists $\kappa^0 \in \mathcal{H}(\theta)$ such that*

(1) $\varpi_F \in \operatorname{Ker}\kappa^0$ *and*
(2) $C_E(\varpi_F) \subset \operatorname{Ker}\det\kappa^0$.

These conditions determine κ^0 uniquely, up to tensoring with a character $\phi \in X_1(\theta)$, trivial on J^0 and such that $\phi^{p^r} = 1$. Moreover, $\boldsymbol{\mu}_F \subset \operatorname{Ker}\kappa^0$.

Proof. The existence of κ^0 follows from 3.2 Proposition, and its uniqueness property from Lemma 1. The group $\boldsymbol{\mu}_F$ is central in G. It has order relatively prime to p, while $\dim\kappa^0$ is a power of p. The final assertion thus follows from condition (2). $\square$

Lemma 3. *Let $\kappa^0 \in \mathcal{H}(\theta)$ satisfy the conditions of* Lemma 2 *relative to the element ϖ_F. There exists a unique $\kappa_K^0 \in \mathcal{H}(\theta_K)$ such that*

(1) $\varpi_F \in \operatorname{Ker} \kappa_K^0$,
(2) $C_E(\varpi_F) \subset \operatorname{Ker} \det \kappa_K^0$, *and*
(3) *if $h \in {}_p\boldsymbol{J}_K(\varpi_F)$, then*

$$\operatorname{tr} \kappa_K^0(h) = \epsilon^0_{\boldsymbol{\mu}_K}\, \epsilon^1_{\boldsymbol{\mu}_K}(\zeta) \operatorname{tr} \kappa^0(\zeta h), \tag{5.6.1}$$

for every $\varGamma$-regular element ζ of $\boldsymbol{\mu}_K$.

Proof. The Glauberman correspondence, together with (5.4.1), yields a unique representation κ_K^0 of ${}_p\boldsymbol{J}_K$ satisfying the desired relation (5.6.1) and condition (1). This representation admits extension to a representation of $\boldsymbol{J}_K = P^\times J_K^1$, and the extension may be chosen to satisfy condition (2). In this case, any representation of $\boldsymbol{J}_K$ extending η_K lies in $\mathcal{H}(\theta_K) = \mathcal{T}(\theta_K)$, so the uniqueness follows from Lemma 1. $\square$

Remark. As in Lemma 2, condition (2) of Lemma 3 implies $\boldsymbol{\mu}_K \subset \operatorname{Ker} \kappa_K^0$. In (5.6.1) therefore, we have $\operatorname{tr} \kappa_K^0(\zeta h) = \operatorname{tr} \kappa_K^0(h)$.

The set $\mathcal{H}(\theta_K)$ carries an action of $X_1(E)$, making it a principal homogeneous space over $X_1(E)$, as in 3.2 Corollary. Consequently, it carries an action of $X_1(E_0)$ via the canonical map $X_1(E_0) \to X_1(E)$ given by $\chi \mapsto \chi_E = \chi \circ \mathrm{N}_{E/E_0}$. If we view $\varDelta$ as $\mathrm{Gal}(E/E_0)$, this canonical map induces an isomorphism $X_1(E_0)/X_0(E_0)_m$ with the group $X_1(E)^\varDelta$ of $\varDelta$-fixed points in $X_1(E)$.

We consider the group $\varDelta = \mathrm{Gal}(P/P_0)$. For each $\delta \in \varDelta$, there exists $j_\delta \in G$ such that $j_\delta^{-1} x j_\delta = x^\delta$, $x \in P^\times$. The same relation holds for $x \in K^\times$ or $E^\times$. The definition of P shows we may choose $j_\delta \in U_{\mathfrak{a}_P} \subset J_\theta^0$. We use these elements j_δ to extend the canonical action of $\varDelta$ on K to one on G_K, in the standard manner. Since $j_\delta \in J^0$, it conjugates θ to itself and therefore normalizes $\boldsymbol{J}_K = \boldsymbol{J}_\theta \cap G_K$. Moreover, conjugation by j_δ fixes θ_K and so $\varDelta$ acts on the set $\mathcal{H}(\theta_K)$ via $\kappa \mapsto \kappa^\delta = \kappa^{j_\delta}$. If $\phi \in X_1(E)$, we also have

$$(\phi \odot \lambda)^\delta \cong \phi^\delta \odot \lambda^\delta, \quad \phi \in X_1(E),\ \lambda \in \mathcal{H}(\theta_K), \tag{5.6.2}$$

where $\varDelta$, viewed as $\mathrm{Gal}(E/E_0)$, acts on $X_1(E)$ in the natural way.

Lemma 4.

(1) *The representation* κ_K^0 *of* Lemma 3 *lies in* $\mathcal{H}(\theta_K)^\Delta$, *and*
(2) $\mathcal{H}(\theta_K)^\Delta$ *is a principal homogeneous space over* $X_1(E_0)/X_0(E_0)_m$.

Proof. Since $j_\delta \in J^0$, the function $\operatorname{tr}\kappa^0$ is invariant under conjugation by j_δ. Assertion (1) now follows from the definition of κ_K^0, and (2) from (5.6.2). $\square$

Proposition. *Let* $\kappa^0 \in \mathcal{H}(\theta)$ *satisfy the conditions of* Lemma 2, *and define* $\kappa_K^0 \in \mathcal{H}(\theta_K)$ *as in* Lemma 3.

(1) *There is a unique* $X_1(E_0)$*-map*

$$\ell_{K/F} : \mathcal{H}(\theta) \longrightarrow \mathcal{H}(\theta_K) \tag{5.6.3}$$

such that

$$\ell_{K/F}(\kappa^0) = \kappa_K^0. \tag{5.6.4}$$

(2) *The map* $\ell_{K/F}$ *is injective, and its image is the set* $\mathcal{H}(\theta_K)^\Delta$ *of* Δ*-fixed points in* $\mathcal{H}(\theta_K)$.
(3) *The definition of* $\ell_{K/F}$ *is independent of the choices of* ϖ_F *and the representation* κ^0 *satisfying the conditions of* Lemma 2, *relative to* ϖ_F.

Proof. Parts (1) and (2) follow from the fact that $\mathcal{H}(\theta)$ and $\mathcal{H}(\theta_K)^\Delta$ are both principal homogeneous spaces over $X_1(E_0)/X_0(E_0)_m$.

We prove part (3). We work at first relative to fixed choices of ϖ_F and κ^0. Taking $\kappa \in \mathcal{H}(\theta)$, we write $\kappa = \phi \odot \kappa^0$, for $\phi \in X_1(E_0)$ uniquely determined modulo $X_0(E_0)_m$ (*cf.* 3.2 Corollary). We define

$$\ell_{K/F}(\kappa) = \ell_{K/F}(\phi \odot \kappa^0) = \phi_E \odot \kappa_K^0.$$

This is the unique $X_0(E_0)$-map $\mathcal{H}(\theta) \to \mathcal{H}(\theta_K)$ with the property (5.6.4). It is clearly injective with image $\mathcal{H}(\theta_K)^\Delta$.

We have to check that this definition of $\ell_{K/F}$ is independent of choices. First, if we keep ϖ_F fixed, the definition is independent of the choice of κ^0, as follows from the uniqueness statement in Lemma 2.

Next, let us replace ϖ_F by $u\varpi_F$, for some $u \in U_F^1$. The group ${}_p\boldsymbol{J}(\varpi_F)$ is then unchanged, but C_E is replaced by $C'_E = \langle v\varpi_{E_0}, \boldsymbol{\mu}_K\rangle$, where $v \in U_F^1$ satisfies

$v^e = u$. There is a character $\phi \in X_0(E_0)$, of p-power order modulo $X_0(E_0)_m$, such that $\phi \odot \kappa^0$ satisfies the conditions of Lemma 2 relative to ϖ'_F. The element of $\mathcal{H}(\theta_K)$ corresponding to $\phi \odot \kappa^0$ via Lemma 3 is $\phi_E \odot \kappa^0_K = \ell_{K/F}(\phi \odot \kappa^0)$, as required.

Suppose now that ϖ_F is replaced by $\alpha\varpi_F$, for some $\alpha \in \boldsymbol{\mu}_F$. The group C_E is then unchanged. If the infinite cyclic group ${}_p\boldsymbol{J}(\varpi_F)/J^1$ is generated by ϖ_0, then ${}_p\boldsymbol{J}(\alpha\varpi_F)/J^1$ is generated by $\alpha_1\varpi_0$, where $\alpha_1 \in \boldsymbol{\mu}_F$ satisfies $\alpha_1^{p^r} = \alpha$. However, we have remarked that $\boldsymbol{\mu}_F \subset \operatorname{Ker}\kappa^0$. In particular, $\alpha \in \operatorname{Ker}\kappa^0$ and the same representation κ^0 therefore satisfies the conditions of Lemma 2 relative to $\alpha\varpi_F$. Let $h \in {}_p\boldsymbol{J}_K(\alpha\varpi_F)$; there exists $\varepsilon \in \boldsymbol{\mu}_F$ such that $\varepsilon h \in {}_p\boldsymbol{J}_K(\varpi_F)$. The relation (5.6.1) reads

$$\begin{aligned}\epsilon^0_{\boldsymbol{\mu}_K}\,\epsilon^1_{\boldsymbol{\mu}_K}(\zeta)\operatorname{tr}\kappa^0(\zeta h) &= \epsilon^0_{\boldsymbol{\mu}_K}\,\epsilon^1_{\boldsymbol{\mu}_K}(\zeta\varepsilon)\operatorname{tr}\kappa^0(\zeta\varepsilon h)\\ &= \operatorname{tr}\kappa^0_K(\zeta\varepsilon h) = \operatorname{tr}\kappa^0_K(h) = \operatorname{tr}\kappa^0_K(\zeta h),\end{aligned}$$

since κ^0_K is trivial on $\boldsymbol{\mu}_K$ and κ^0, $\epsilon^1_{\boldsymbol{\mu}_K}$ are both trivial on $\boldsymbol{\mu}_F$. In other words, the representation κ^0_K is also unchanged and the result follows. □

5.7. We consider the set $\mathcal{H}(\theta_K)^{\Delta\text{-reg}}$ of *Δ-regular* elements of $\mathcal{H}(\theta_K) = \mathcal{T}(\theta_K)$.

Lemma 1. *A representation $\Lambda \in \mathcal{H}(\theta_K)$ is Δ-regular if and only if it is equivalent to $\xi \odot \ell_{K/F}(\kappa)$, for some $\kappa \in \mathcal{H}(\theta)$ and a Δ-regular element ξ of $X_1(E)$.*

Proof. This follows from 5.6 Proposition and (5.6.2). □

Let $X_1(E)^{\Delta\text{-reg}}$ denote the set of Δ-regular elements of $X_1(E)$. A character $\xi \in X_1(E)^{\Delta\text{-reg}}$ determines a representation $\lambda^{\boldsymbol{J}}_\xi$ of $\boldsymbol{J}/J^1$ as in 3.6. In the notation of 3.6, we may form the representation $\lambda^{\boldsymbol{J}}_\xi \otimes \kappa = \lambda_\xi \ltimes \kappa \in \mathcal{T}(\theta)$, for any $\kappa \in \mathcal{H}(\theta)$.

Lemma 2. *For any $\kappa \in \mathcal{H}(\theta)$, the map $\xi \mapsto \lambda_\xi \ltimes \kappa$ induces an $X_1(E_0)$-isomorphism $\Delta\backslash X_1(E)^{\Delta\text{-reg}} \to \mathcal{T}(\theta)$.*

Proof. This follows from Lemma 1 and 3.6 Proposition (2). □

We define a map $ind_{K/F} : \mathcal{T}(\theta_K)^{\Delta\text{-reg}} \to \mathcal{T}(\theta)$ as follows. Choose $\kappa \in \mathcal{H}(\theta)$ and let $\Lambda \in \mathcal{T}(\theta_K)^{\Delta\text{-reg}}$. Using Lemma 1, we may write $\Lambda = \xi \odot \ell_{K/F}(\kappa)$, for some $\xi \in X_1(E)^{\Delta\text{-reg}}$. We set

$$ind_{K/F}\,\Lambda = \lambda_\xi \ltimes \kappa. \tag{5.7.1}$$

The definition is independent of the choice of $\kappa \in \mathcal{H}(\theta)$ (*cf.* 3.6 Proposition (1)). We so obtain a *canonical* bijection

$$ind_{K/F} : \Delta\backslash\mathcal{T}(\theta_K)^{\Delta\text{-reg}} \xrightarrow{\approx} \mathcal{T}(\theta). \tag{5.7.2}$$

By construction, it satisfies

$$ind_{K/F}\,\big(\phi_E \odot \Lambda\big) \cong \phi \odot ind_{K/F}\,\Lambda, \tag{5.7.3}$$

for all $\Lambda \in \mathcal{T}(\theta_K)^{\Delta\text{-reg}}$ and $\phi \in X_1(E_0)$.

Comments. All of the preceding constructions are in terms of a field P/P_0, chosen as in 5.4 Notation (5). The field K/F is then defined as the maximal unramified sub-extension of P/F. As remarked at the time, any two choices of P, hence of K, are J^0_θ-conjugate. Indeed, they are conjugate by an element of J^0_θ commuting with P_0. All of the constructions are invariant under such conjugations: if $j \in J^0_\theta \cap G_{P_0}$, then conjugation by j gives a bijection $\mathcal{H}(\theta_K) \to \mathcal{H}(\theta_{K^j})$ preserving the actions of $X_1(E_0)$. Also, conjugation by j fixes the representations $\lambda^{\boldsymbol{J}}_\xi$ and κ. Thus $ind_{K^j/F}\,\Lambda^j \cong ind_{K/F}\,\Lambda$, and so the constructions are independent of the choices of P and K.

5.8. Following 4.2, we translate this machinery to the context of cuspidal representations of the groups G and G_K. To do this, we must first "externalize" our notation. From this point of view, the hypotheses listed in 5.4 give us:

Data.

(1) *a finite unramified field extension K/F,*
(2) *an endo-class $\Theta_K \in \mathcal{E}(K)$ with totally ramified parameter field E/K,*
(3) *a group Δ of F-automorphisms of E, of order m and such that E/E^Δ is unramified and*
(4) *a totally wild E/K-lift Θ_E of Θ_K*

such that

(5) *$E_0 = E^\Delta/F$ is a tame parameter field for Θ.*

In particular, $\Theta_{E_0} = \mathfrak{i}_{E/E_0}\Theta_E$ is a totally wild E_0/F-lift of Θ and Θ_E is the unique E/E_0-lift of Θ_{E_0}. Thus Θ_E is fixed by Δ. We identify Δ with $\mathrm{Gal}(K/K \cap E_0)$ by restriction, and Δ then fixes Θ_K.

Set $n = m \deg \varTheta$ and $n_K = n/[K{:}F]$. Let $G_K = \mathrm{GL}_{n_K}(K)$. We choose an m-realization θ_K of $\varTheta_K$ in G_K. If T/K is a tame parameter field for θ_K, we choose a K-isomorphism $\boldsymbol{\iota} : E \to T$ such that $\boldsymbol{\iota}_*(\varTheta_K) = cl(\theta_K)$. This configuration is determined up to conjugation by an element of the G_K-centralizer $G_{\boldsymbol{\iota}E}$ of $\boldsymbol{\iota}E^\times$ (as in 2.7). As in (4.1.3), we have the bijection $\mathcal{T}(\theta_K) \to \mathcal{A}_1^0(K;\varTheta_K)$ given by $\varLambda \mapsto c\text{-}\mathrm{Ind}_{\boldsymbol{J}_{\theta_K}}^{G_K} \varLambda$. This map takes the natural $\odot$-action of $X_1(E)$ on $\mathcal{H}(\theta_K) = \mathcal{T}(\theta_K)$ to its $\odot_{\varTheta_E}$-action on $\mathcal{A}_1^0(K;\varTheta_K)$.

Next, we follow (2.3.3) to produce a simple character θ in $G = \mathrm{GL}_n(F)$ such that $H_\theta^1 \cap G_K = H_{\theta_K}^1$ and $\theta|_{H^1_{\theta_K}} = \theta_K$. Thus θ is an m-realization of $\varTheta$, with tame parameter field $\boldsymbol{\iota}E_0/F$. The character θ_K determines the character θ uniquely [3] 7.15. We view $\varDelta$ as acting on G_K via conjugation by elements of J_θ^0, as in 5.6. This induces the natural action of $\varDelta$ on $\mathcal{A}_1^0(K;\varTheta_K)$ and the induction map $\mathcal{T}(\theta_K) \to \mathcal{A}_1^0(K;\varTheta_K)$ is a $\varDelta$-map. We likewise have the induction map $\mathcal{T}(\theta) \to \mathcal{A}_m^0(F;\varTheta)$, transforming the $\odot$-action of $X_1(E_0)$ on $\mathcal{T}(\theta)$ into its $\odot_{\varTheta_{E_0}}$-action on $\mathcal{A}_m^0(F;\varTheta)$.

Proposition. *The map $ind_{K/F}$ of* (5.7.1) *induces a canonical bijection*

$$ind_{K/F} : \varDelta\backslash\mathcal{A}_1^0(K;\varTheta_K)^{\varDelta\text{-reg}} \xrightarrow{\approx} \mathcal{A}_m^0(F;\varTheta) \tag{5.8.2}$$

satisfying

$$ind_{K/F}\big(\phi_E \odot_{\varTheta_E} \tau\big) = \phi \odot_{\varTheta_{E_0}} ind_{K/F}\,\tau, \tag{5.8.3}$$

for $\tau \in \mathcal{A}_1^0(K;\varTheta_K)^{\varDelta\text{-reg}}$ and $\phi \in X_1(E_0)$.

If $\gamma : K \to K^\gamma$ is an isomorphism of local fields, then

$$ind_{K^\gamma/F^\gamma}\,\tau^\gamma \cong \big(ind_{K/F}\,\tau\big)^\gamma, \tag{5.8.4}$$

for all $\tau \in \mathcal{A}_1^0(K;\varTheta_K)^{\varDelta\text{-reg}}$.

Proof. The triple $(\theta_K, \boldsymbol{\iota})$ is determined up to conjugation by an element of $G_{\boldsymbol{\iota}E}$, as already noted. Since θ is determined by θ_K, the triple $(\theta_K, \boldsymbol{\iota}, \theta)$ is likewise uniquely determined, up to conjugation by $G_{\boldsymbol{\iota}E}$. The map $\mathcal{A}_1^0(K;\varTheta_K)^{\varDelta\text{-reg}} \to \mathcal{A}_m^0(F;\varTheta)$, induced by the map $ind_{K/F} : \mathcal{T}(\theta_K)^{\varDelta\text{-reg}} \to \mathcal{T}(\theta)$, is therefore independent of the choice of $(\theta_K, \boldsymbol{\iota}, \theta)$.

That the map $ind_{K/F}$ of (5.7.1) induces a bijection (5.8.2) follows from (5.7.2). The property (5.8.3) follows from (5.7.3).

Finally, consider the map $\Lambda \mapsto \big(ind_{K^\gamma/F^\gamma}\,\Lambda^\gamma\big)^{\gamma^{-1}}$, $\Lambda \in \mathcal{T}(\theta_K)^{\Delta\text{-reg}}$. This is a version of $ind_{K/F}$ defined relative to different choices of $(\theta_K, \boldsymbol{\iota}, \theta)$. Such choices have no effect on the map (5.8.1), whence (5.8.4) follows. □

5.9. We continue in the same situation, as laid out at the beginning of 5.4 but using the viewpoint of 5.8.

The group Δ acts on E as $\mathrm{Gal}(E/E_0)$. It fixes the endo-class Θ_E, so it acts on $\mathcal{A}_1^0(E;\Theta_E)$. Let $\mathcal{A}_1^0(E;\Theta_E)^{\Delta\text{-reg}}$ be the subset of Δ-regular elements. From (5.3.6), applied with base field K, we get a canonical bijection $ind_{E/K}$: $\mathcal{A}_1^0(E;\Theta_E) \to \mathcal{A}_1^0(K;\Theta_K)$. By 5.3 Corollary 2, this is a Δ-map. We therefore define a map

$$ind_{E/F} : \mathcal{A}_1^0(E;\Theta_E)^{\Delta\text{-reg}} \longrightarrow \mathcal{A}_m^0(F;\Theta)$$

by

$$ind_{E/F} = ind_{K/F} \circ ind_{E/K}. \tag{5.9.1}$$

We combine 5.3 Corollary 1 with 5.8 Proposition to obtain:

Theorem. *Let $\Theta \in \mathcal{E}(F)$ have tame parameter field E_0/F. Let E/E_0 be unramified of degree m, and set $\Delta = \mathrm{Gal}(E/E_0)$. Let Θ_{E_0} be a totally wild E_0/F-lift of Θ, let Θ_E be the unique E/E_0-lift of Θ_{E_0}. The map*

$$ind_{E/F} : \Delta\backslash\mathcal{A}_1^0(E;\Theta_E)^{\Delta\text{-reg}} \longrightarrow \mathcal{A}_m^0(F;\Theta)$$

is a canonical bijection satisfying

$$ind_{E/F}(\phi_E\,\pi) = \phi \odot_{\Theta_{E_0}} ind_{E/F}\,\pi,$$

for $\pi \in \mathcal{A}_1^0(E;\Theta_E)^{\Delta\text{-reg}}$, $\phi \in X_1(E_0)$, and $\phi_E = \phi \circ \mathrm{N}_{E/E_0}$.

The map $ind_{E/F}$ is natural with respect to isomorphisms of the field E.

6. Some properties of the Langlands correspondence

We make some preliminary connections between the machinery developed in §2–§4 and the representation theory of the Weil group set out in §1. We

use the same notation as before. In particular, $\mathcal{G}_n^0(F)$ is the set of equivalence classes of irreducible, smooth, n-dimensional representations of $\mathcal{W}_F$ and $\widehat{\mathcal{W}}_F = \bigcup_{n\geqslant 1} \mathcal{G}_n^0(F)$. It will be convenient to have the analogous notation $\widehat{\mathrm{GL}}_F = \bigcup_{n\geqslant 1} \mathcal{A}_n^0(F)$. The Langlands correspondence therefore gives a bijection

$$\begin{aligned} \mathbb{L} : \widehat{\mathcal{W}}_F &\longrightarrow \widehat{\mathrm{GL}}_F, \\ \sigma &\longmapsto {}^L\sigma. \end{aligned}$$

6.1. To a representation $\pi \in \widehat{\mathrm{GL}}_F$ we attach the endo-class $\vartheta(\pi) \in \mathcal{E}(F)$ of an m-simple character occurring in π, as in (4.1.1). On the other hand, let

$$r_F^1 : \widehat{\mathcal{W}}_F \longrightarrow \mathcal{W}_F \backslash \widehat{\mathcal{P}}_F \tag{6.1.1}$$

be the map taking an irreducible smooth representation σ of $\mathcal{W}_F$ to the $\mathcal{W}_F$-orbit $\mathcal{O}_F(\alpha)$ of an irreducible component α of $\sigma|_{\mathcal{P}_F}$.

Ramification Theorem. *There is a unique map*

$$\Phi_F : \mathcal{W}_F \backslash \widehat{\mathcal{P}}_F \longrightarrow \mathcal{E}(F)$$

such that the following diagram commutes.

$$\begin{array}{ccc} \widehat{\mathcal{W}}_F & \xrightarrow{\ \mathbb{L}\ } & \widehat{\mathrm{GL}}_F \\ {\scriptstyle r_F^1}\big\downarrow & & \big\downarrow{\scriptstyle \vartheta} \\ \mathcal{W}_F \backslash \widehat{\mathcal{P}}_F & \xrightarrow[\ \Phi_F\ ]{} & \mathcal{E}(F) \end{array}$$

The map Φ_F is bijective. If $\gamma : F \to F^\gamma$ is an isomorphism of topological fields, extended in some way to an isomorphism $\mathcal{W}_F \to \mathcal{W}_{F^\gamma}$, then

$$\Phi_{F^\gamma}(\mathcal{O}_{F^\gamma}(\alpha^\gamma)) = \Phi_F(\mathcal{O}_F(\alpha))^\gamma. \tag{6.1.2}$$

Proof. All assertions except the last are 8.2 Theorem of [8]. The last one follows from the uniqueness of Φ_F and the corresponding property of the Langlands correspondence. $\square$

If $\alpha \in \widehat{\mathcal{P}}_F$, we usually write $\Phi_F(\alpha)$ rather than $\Phi_F(\mathcal{O}_F(\alpha))$.

When applied to one-dimensional representations of $\mathcal{P}_F$, the Ramification Theorem reduces to the standard ramification theorem of local class field theory.

6.2. If K/F is finite and tamely ramified, then $\mathcal{P}_K = \mathcal{P}_F$ and there is an obvious surjective map

$$\boldsymbol{i}_{K/F} : \mathcal{W}_K\backslash\widehat{\mathcal{P}}_F \longrightarrow \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F, \qquad \mathcal{O}_K(\alpha) \longmapsto \mathcal{O}_F(\alpha). \tag{6.2.1}$$

We also have the map $\mathrm{i}_{K/F} : \mathcal{E}(K) \to \mathcal{E}(F)$ of (2.3.4), such that $\mathrm{i}_{K/F}^{-1}\Theta$ is the set of K/F-lifts of $\Theta \in \mathcal{E}(F)$.

Proposition. *Let K/F be a finite, tamely ramified field extension with $K \subset \bar{F}$. The diagram*

$$\begin{array}{ccc} \mathcal{W}_K\backslash\widehat{\mathcal{P}}_K & \xrightarrow{\Phi_K} & \mathcal{E}(K) \\ {\scriptstyle \boldsymbol{i}_{K/F}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{i}_{K/F}} \\ \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F & \xrightarrow[\Phi_F]{} & \mathcal{E}(F) \end{array}$$

is commutative.

Proof. The assertion is transitive with respect to the finite tame extension K/F. We may therefore assume that K/F is *of prime degree.*

We deal first with the case where K/F is *cyclic.* Let $\alpha \in \widehat{\mathcal{P}}_F$, and let τ be an irreducible representation of $\mathcal{W}_K$ such that $\tau|_{\mathcal{P}_F}$ contains α, that is, $r_K^1(\tau) = \mathcal{O}_K(\alpha)$. Set $\phi = {}^L\tau$, so that $\vartheta(\phi) = \Phi_K(\alpha)$ by the Ramification Theorem. Consider the representation $\sigma = \operatorname{Ind}_{K/F}\tau$.

Suppose σ is not irreducible: equivalently, $\tau^\gamma \cong \tau$ for all $\gamma \in \mathit{\Gamma} = \mathrm{Gal}(K/F)$. We have

$$\sigma = \bigoplus_{\chi} \chi \otimes \sigma_1,$$

where σ_1 is irreducible and χ ranges over the characters of the group $\mathcal{W}_F/\mathcal{W}_K = \mathit{\Gamma}$. All characters χ appearing here are trivial on $\mathcal{P}_K = \mathcal{P}_F$, so $r_F^1(\sigma_1)$ is the orbit $\mathcal{O}_F(\alpha) = \boldsymbol{i}_{K/F}\mathcal{O}_K(\alpha)$. On the other hand, the Langlands correspondence takes σ to the representation $\pi = \mathrm{A}_{K/F}(\phi)$ automorphically induced by ϕ [16] 3.1, and this takes the form of a Zelevinsky sum [30]

$$\pi = \boxplus_{\chi} \chi\pi_1,$$

where $\pi_1 = {}^L\sigma_1$ and χ ranges as before. We have

$$\vartheta(\pi_1) = \Phi_F(r_F^1(\sigma_1)) = \Phi_F(\mathcal{O}_F(\alpha)) = \Phi_F \circ \boldsymbol{i}_{K/F}(\mathcal{O}_K(\alpha))$$

while, according to 7.1 Corollary of [8],

$$\vartheta(\pi_1) = \mathfrak{i}_{K/F}(\vartheta(\phi)) = \mathfrak{i}_{K/F}(\Phi_K(r^1_K(\tau))) = \mathfrak{i}_{K/F} \circ \Phi_K(\mathcal{O}_K(\alpha)).$$

Therefore

$$\Phi_F \circ \boldsymbol{i}_{K/F}(\mathcal{O}_K(\alpha)) = \vartheta(\pi_1) = \mathfrak{i}_{K/F} \circ \Phi_K(\mathcal{O}_K(\alpha)),$$

which proves the proposition in the present case.

Suppose now that σ is irreducible. The representation $\pi = {}^L\sigma$ is again automorphically induced, $\pi = \mathrm{A}_{K/F}(\phi)$ and, in the same way, $\vartheta(\pi) = \mathfrak{i}_{K/F}(\vartheta(\phi))$. The result follows as before.

The proposition thus holds when K/F is cyclic of prime degree. By transitivity, it holds when the tame extension K/F is Galois of any finite degree.

We are reduced to the case where K/F is of prime degree but not cyclic. In particular, K/F is totally tamely ramifed. Let E/F be the normal closure of K/F. From the first part of the proof, we know that the result holds for the tamely ramified Galois extensions E/F, E/K. A diagram chase shows that it holds for K/F. $\square$

6.3. We use 6.2 Proposition to refine the Ramification Theorem. As in (1.5.1), we set

$$d_F(\alpha) = [Z_F(\alpha){:}F]\ \dim\alpha, \quad \alpha \in \widehat{\mathcal{P}}_F.$$

Tame Parameter Theorem. *Let $\alpha \in \widehat{\mathcal{P}}_F$ have dimension p^r, $r \geqslant 0$. If $E = Z_F(\alpha)$, then*

(1) $\deg \Phi_F(\alpha) = d_F(\alpha)$, *and*
(2) *E/F is a tame parameter field for $\Phi_F(\alpha)$.*

Proof. We recall from [17] 6.2.5 the following relation.

Lemma 1. *Let $\pi \in \mathcal{A}^0_n(F)$, and denote by $t(\pi)$ the number of characters $\chi \in X_0(F)$ for which $\chi\pi \cong \pi$. We then have $t(\pi) = n/e(\vartheta(\pi))$.*

Similarly, if $\sigma \in \widehat{\mathcal{W}}_F$, we define $t(\sigma)$ to be the number of $\chi \in X_0(F)$ such that $\chi \otimes \sigma \cong \sigma$. Since ${}^L(\chi \otimes \sigma) = \chi \cdot {}^L\sigma$, we have

$$t(\sigma) = t({}^L\sigma). \tag{6.3.1}$$

Using 1.4 Theorem to write $\sigma = \Sigma(\rho,\tau)$, for an admissible datum $(E/F,\rho,\tau)$, we find

$$t(\sigma) = f(E|F)\,\dim\tau. \tag{6.3.2}$$

We apply Lemma 1, first in the case where $E = F$. By 1.3 Proposition, there exists $\sigma \in \widehat{\mathcal{W}}_F$ such that $\sigma|_{\mathcal{P}_F} \cong \alpha$. The representation σ satisfies $t(\sigma) = 1$ (6.3.2) and $\dim\sigma = \dim\alpha = p^r$. If $\pi = {}^L\sigma$, it follows that $t(\pi) = 1$ and then that $e(\vartheta(\pi)) = \dim\sigma = p^r$. We deduce that $\deg\vartheta(\pi) = p^r$ and, if $F[\beta]$ is a parameter field for $\vartheta(\pi)$, then $F[\beta]/F$ is totally wildly ramified. The tame parameter field for $\vartheta(\pi) = \Phi_F(\alpha)$ is therefore F, as required.

We pass to the general case $E \neq F$.

Lemma 2. *The field E contains a tame parameter field T/F for $\Phi_F(\alpha)$, and* $\deg\Phi_F(\alpha) = p^r[T{:}F]$.

Proof. According to 6.2 Proposition, we have $\Phi_F(\alpha) = \mathrm{i}_{E/F}\Phi_E(\alpha)$. The fibre $\boldsymbol{i}_{E/F}^{-1}(\mathcal{O}_F(\alpha))$ is given by the disjoint union

$$\boldsymbol{i}_{E/F}^{-1}\mathcal{O}_F(\alpha) = \bigcup_{g\in\mathcal{W}_E\backslash\mathcal{W}_F/\mathcal{W}_E} \mathcal{O}_E(\alpha^g).$$

By the first case above, the term $\Phi_E(\alpha)$ is totally wild of degree p^r, but is also an E/F-lift of $\Phi_F(\alpha)$. The lemma now follows from 2.4 Proposition. $\square$

Set $s = p^r[T{:}F]$. There exists an irreducible cuspidal representation τ of $\mathrm{GL}_s(F)$ with $\vartheta(\tau) = \Phi_F(\alpha)$. Define $\nu \in \widehat{\mathcal{W}}_F$ by ${}^L\nu = \tau$. Thus $\dim\nu = s$ while $\nu|_{\mathcal{P}_F}$ contains the representation α. However, an irreducible smooth representation of $\mathcal{W}_F$ containing α has dimension divisible by $p^r[E{:}F]$ (1.4 Theorem). Therefore $s = p^r[E{:}F]$. By the lemma, $T \subset E$, so $E = T$. $\square$

6.4. Let $\mathcal{O} \in \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$, let $\varTheta \in \mathcal{E}(F)$ and let $m \geqslant 1$ be an integer. We define sets $\mathcal{G}_m^0(F;\mathcal{O})$, $\mathcal{A}_m^0(F;\varTheta)$ as in 1.5, (4.1.2) respectively.

If $\alpha \in \widehat{\mathcal{P}}_F$ and $\mathcal{O} = \mathcal{O}_F(\alpha)$, we set $d(\mathcal{O}) = d_F(\alpha)$. Thus $\mathcal{G}_m^0(F;\mathcal{O}) \subset \mathcal{G}_n^0(F)$, where $n = md(\mathcal{O})$, and $\mathcal{G}_n^0(F)$ is the disjoint union

$$\mathcal{G}_n^0(F) = \bigcup_{(\mathcal{O},m)} \mathcal{G}_m^0(F;\mathcal{O}), \tag{6.4.1}$$

where $(\mathcal{O}, m)$ ranges over all pairs $\mathcal{O} \in \mathcal{W}_F \backslash \widehat{\mathcal{P}}_F$, $m \in \mathbb{Z}$, such that $n = md(\mathcal{O})$. Likewise, $\mathcal{A}_n^0(F)$ is the disjoint union

$$\mathcal{A}_n^0(F) = \bigcup_{(\Theta,m)} \mathcal{A}_m^0(F;\Theta), \tag{6.4.2}$$

where $\Theta \in \mathcal{E}(F)$, $m \in \mathbb{Z}$ and $n = m \deg \Theta$.

Corollary. *Let $\mathcal{O} \in \mathcal{W}_F \backslash \widehat{\mathcal{P}}_F$ and put $\Theta = \Phi_F(\mathcal{O}) \in \mathcal{E}(F)$. For every integer $m \geqslant 1$, the Langlands correspondence induces a bijection*

$$\mathcal{G}_m^0(F;\mathcal{O}) \xrightarrow{\approx} \mathcal{A}_m^0(F;\Theta). \tag{6.4.3}$$

Proof. Let $\rho \in \mathcal{G}_m^0(F;\mathcal{O})$. By the Ramification Theorem, ${}^L\rho \in \mathcal{A}_{m'}^0(F;\Theta)$, where $m' \deg \Theta = md(\mathcal{O})$. Part (1) of the Tame Parameter Theorem says $\deg \Theta = d(\mathcal{O})$. Consequently

$${}^L\big(\mathcal{G}_m^0(F;\mathcal{O})\big) \subset \mathcal{A}_m^0(F;\Phi_F(\mathcal{O})), \quad \mathcal{O} \in \mathcal{W}_F \backslash \widehat{\mathcal{P}}_F,\ m \geqslant 1.$$

The corollary now follows from (6.4.1), (6.4.2). □

7. A naïve correspondence and the Langlands correspondence

We use the machinery of §5, and the relationships uncovered in §6, to define a canonical bijection

$$\begin{aligned} \mathbb{N} : \widehat{\mathcal{W}}_F &\longrightarrow \widehat{\mathrm{GL}}_F, \\ \sigma &\longmapsto {}^N\sigma. \end{aligned}$$

We state our main results, comparing this "naïve correspondence" with the Langlands correspondence.

7.1. Suppose first that $\sigma \in \widehat{\mathcal{W}}_F$ is *totally wildly ramified,* i.e., $\sigma|_{\mathcal{P}_F}$ is irreducible. For such a representation σ, we set

$${}^N\sigma = {}^L\sigma. \tag{7.1.1}$$

Basic properties of the Langlands correspondence [16] 3.1 imply:

Proposition. *Let $\sigma \in \widehat{\mathcal{W}}_F$ be totally wildly ramified.*

(1) *If χ is a character of $F^\times$, then*

$$ {}^N(\chi \otimes \sigma) = \chi \cdot {}^N\sigma. \tag{7.1.2} $$

(2) *If $\gamma : F \to F^\gamma$ is an isomorphism of local fields, then*

$$ {}^N(\sigma^\gamma) = ({}^N\sigma)^\gamma, \tag{7.1.3} $$

7.2. Let $\sigma \in \widehat{\mathcal{W}}_F$. As in 6.4, there is a unique pair $(m, \mathcal{O})$, where $m \geqslant 1$ and $\mathcal{O} \in \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$, such that $\sigma \in \mathcal{G}^0_m(F; \mathcal{O})$. Choose $\alpha \in \mathcal{O}$, and set $E = Z_F(\alpha)$. Let E_m/E be unramified of degree m and put $\varDelta = \mathrm{Gal}(E_m/E)$. As in 1.6 Proposition, there is a $\varDelta$-regular representation $\rho \in \mathcal{G}^0_1(E_m; \alpha)$ such that $\sigma \cong \mathrm{Ind}_{E_m/F}\,\rho$. The $\varDelta$-orbit of ρ is thereby uniquely determined.

By (7.1.3) and 6.4 Corollary, the representation ${}^N\rho$ is a $\varDelta$-regular element of $\mathcal{A}^0_1(E_m; \Phi_{E_m}(\alpha))$. We use the map $ind_{E_m/F}$ of (5.9.1) to define

$$ {}^N\sigma = ind_{E_m/F}\,{}^N\rho. \tag{7.2.1} $$

Proposition. *For each integer $m \geqslant 1$ and each $\mathcal{O} \in \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$, the assignment* (7.2.1) *induces a bijection*

$$ \begin{gathered} \mathcal{G}^0_m(F; \mathcal{O}) \longrightarrow \mathcal{A}^0_m(F; \Phi_F(\mathcal{O})), \\ \sigma \longmapsto {}^N\sigma, \end{gathered} \tag{7.2.2} $$

which does not depend on the choice of $\alpha \in \mathcal{O}$ used in its definition. Further,

(1) *if $x \mapsto x^\gamma$ is an isomorphism $F \to F^\gamma$ of local fields, then*

$$ {}^N(\sigma^\gamma) = \big({}^N\sigma\big)^\gamma, \quad \sigma \in \mathcal{G}^0_m(F; \mathcal{O}), $$

and

(2) *if $\alpha \in \mathcal{O}$ and $E = Z_F(\alpha)$, then*

$$ {}^N(\phi \odot_\alpha \sigma) = \phi \odot_{\Phi_E(\alpha)} {}^N\sigma, $$

for all $\sigma \in \mathcal{G}^0_m(F; \mathcal{O})$ and all $\phi \in X_1(E)$.

Proof. The map $\mathrm{Ind}_{E_m/F}$ induces a canonical bijection $\Delta\backslash\mathcal{G}_1^0(E_m;\alpha)^{\Delta\text{-reg}} \to \mathcal{G}_m^0(F;\mathcal{O})$ (1.6 Proposition). The map $ind_{E_m/F}$ induces a canonical bijection $\Delta\backslash\mathcal{A}_1^0(E_m;\Phi_{E_m}(\alpha))^{\Delta\text{-reg}} \to \mathcal{A}_m^0(F;\Phi_F(\alpha))$ (5.9 Theorem). The naïve correspondence induces a Δ-bijection $\mathcal{G}_1^0(E_m;\alpha) \to \mathcal{A}_1^0(E_m;\Phi_{E_m}(\alpha))$ (7.1 Proposition). The map (7.2.2) is therefore a bijection. If $\alpha' \in \mathcal{O}$, there exists $\gamma \in \mathcal{W}_F$ such that $\alpha' = \alpha^\gamma$, whence (by 5.9 Theorem, (6.1.2) and 7.1 Proposition) the definition of ${}^N\sigma$ does not depend on α. Assertions (1) and (2) likewise follow from 5.9 Theorem. $\square$

In the notation of the proposition, the central character ω_π of $\pi = {}^L\sigma$ is given by

$$\omega_\pi = \det\sigma. \tag{7.2.3}$$

If we write $\sigma = \mathrm{Ind}_{E_m/F}\,\rho$, where $\rho \in \mathcal{G}_1^0(E_m;\alpha)$ is Δ-regular, then

$$\det\sigma = (d_{E_m/F})^{p^r}\,\det\rho|_{F^\times}, \tag{7.2.4}$$

where $d_{E_m/F} \in X_1(F)$ is the discriminantal character of the extension E_m/F,

$$d_{E_m/F} = \det\mathrm{Ind}_{E_m/F}\,1_{E_m},$$

and $p^r = \dim\rho = \dim\alpha$. On the other hand, setting $\pi' = {}^N\sigma$, we get

$$\omega_{\pi'} = \det\rho|_{F^\times}. \tag{7.2.5}$$

Remark. Suppose that $\sigma \in \widehat{\mathcal{W}}_F$ is *essentially tame,* that is, $r_F^1(\sigma) = \mathcal{O}_F(\alpha)$, where $\dim\alpha = 1$. The definition of ${}^N\sigma$ is then equivalent to that of [10] 2.3 Theorem, the first step (7.1.1) in that case being given by local class field theory.

7.3. We state our main result.

Comparison Theorem. *Let $m \geqslant 1$ be an integer and $\alpha \in \widehat{\mathcal{P}}_F$. Let $E = Z_F(\alpha)$ and put $\mathcal{O} = \mathcal{O}_F(\alpha) \in \mathcal{W}_F\backslash\widehat{\mathcal{P}}_F$. There exists a character $\mu = \mu_{m,\alpha}^F \in X_1(E)$ such that*

$${}^L\sigma = \mu \odot_{\Phi_E(\alpha)} {}^N\sigma, \tag{7.3.1}$$

for all $\sigma \in \mathcal{G}_m^0(F;\mathcal{O})$. The character $\mu_{m,\alpha}^F$ is thereby uniquely determined modulo $X_0(E)_m$.

The proof of the Comparison Theorem will occupy the rest of the paper.

Remark 1. Let $\sigma \in \mathcal{G}_m^0(F;\mathcal{O})$, as in the statement of the theorem. Let E_m/E be unramified of degree m and set $\Delta = \mathrm{Gal}(E_m/E)$ (as in the definition (7.2.1)). The theorem asserts that the character $\mu^F_{m,\alpha} \circ \mathrm{N}_{E_m/E} \in X_1(E_m)^\Delta$ is uniquely determined. It is often more convenient to use this version.

Remark 2. Let $\dim\alpha = p^r$, let $G_E = \mathrm{GL}_{mp^r}(E)$ and let $\det_E : G_E \to E^\times$ be the determinant map. Setting $\pi = {}^L\sigma$ and $\pi' = {}^N\sigma$ as before, (7.2.3–5) yield

$$\mu^F_{m,\alpha} \circ \det{}_E|_{F^\times} = \omega_\pi/\omega_{\pi'} = (d_{E_m/F})^{p^r}.$$

In other words,

$$\mu^F_{m,\alpha}(x)^{mp^r} = d_{E_m/F}(x)^{p^r}, \quad x \in F^\times. \tag{7.3.2}$$

7.4. Combining the Comparison Theorem with 7.2 Proposition, we obtain a valuable corollary.

Homogeneity Theorem. *If $\alpha \in \widehat{\mathcal{P}}_F$ and $E = Z_F(\alpha)$, then*

$${}^L(\phi \odot_\alpha \sigma) = \phi \odot_{\Phi_E(\alpha)} {}^L\sigma,$$

for all $\sigma \in \widehat{\mathcal{W}}_F$ such that $r^1_F(\sigma) = \mathcal{O}_F(\alpha)$ and all $\phi \in X_1(E)$.

Proof. Abbreviating $\mathcal{O} = \mathcal{O}_F(\alpha)$, let $\sigma \in \mathcal{G}^0_m(F;\mathcal{O})$ for some $m \geqslant 1$. Let $\mu = \mu^F_{m,\alpha}$, in the notation of the Comparison Theorem. For $\phi \in X_1(E)$, we have

$$\begin{aligned} {}^L(\phi \odot_\alpha \sigma) &= \mu \odot_{\Phi_E(\alpha)} \phi \odot_{\Phi_E(\alpha)} {}^N\sigma \\ &= \phi \odot_{\Phi_E(\alpha)} \mu \odot_{\Phi_E(\alpha)} {}^N\sigma = \phi \odot_{\Phi_E(\alpha)} {}^L\sigma, \end{aligned}$$

as required. □

In general, the Homogeneity Theorem is weaker than the Comparison Theorem. However, the two are equivalent in the case $m = 1$ since the sets $\mathcal{G}^0_1(F;\mathcal{O})$, $\mathcal{A}^0_1(F;\Phi_F(\alpha))$ are then principal homogeneous spaces over $X_1(E)$ (1.5 Proposition, 4.2 Proposition).

7.5. We comment on a well-known special case [23] or [13] 2.4.

Let $\rho \in \widehat{\mathcal{W}}_F$ be *tamely ramified* of dimension n, that is, $\rho \in \mathcal{G}_n^0(F;\mathbf{1}_F)$. Thus $\rho = \rho_\xi = \operatorname{Ind}_{K/F}\xi$, where K/F is unramified of degree n and $\xi \in X_1(K)$ is K/F-regular. As in 3.5, we form the extended maximal simple type $\lambda_\xi \in \mathcal{T}_n(\mathbf{0}_F)$. The representation ${}^N\rho_\xi$ is then the representation $\pi_\xi \in \mathcal{A}_n^0(F)$ containing λ_ξ.

On the other hand, let χ_2 denote the unramified character of $K^\times$ of order 2. The representation ${}^L\rho_\xi$ is then given by

$$ {}^L\rho_\xi = \pi_{\xi'}, \quad \text{where} \quad \xi' = \chi_2^{n-1}\xi \tag{7.5.1}$$

([13] 2.4 Theorem 2). In the notation of the Comparison Theorem therefore,

$$ \mu^F_{n,\mathbf{1}_F} \circ \mathrm{N}_{K/F} = \chi_2^{n-1}. \tag{7.5.2}$$

7.6. We do not give an explicit formula for the "discrepancy character" $\mu^F_{m,\alpha}$. However, we will be able to describe its restriction to units and so obtain a succinct account of the maximal simple type occurring in ${}^L\sigma$, for any $\sigma \in \widehat{\mathcal{W}}_F$.

We follow the description of representations of the Weil group given by 1.4 Theorem. So, let $(E/F,\rho,\tau)$ be an admissible datum, in which $\dim\tau = m \geqslant 1$, and set $\Sigma = \Sigma(\rho,\tau) = \operatorname{Ind}_{E/F}\rho\otimes\tau$. Following 1.3 Proposition, we may choose the datum $(E/F,\rho,\tau)$ so that $\det\rho|_{\mathcal{I}_E}$ has p-power order. Set $\alpha = \rho|_{\mathcal{P}_F}$, and put $\Theta = \Phi_F(\alpha)$.

We choose an m-realization θ of Θ in $G = \mathrm{GL}_n(F)$, where $n = \dim\Sigma$. We identify E with a tame parameter field for θ, in such a way that the simple character θ_E (as in 2.3) has endo-class $\Phi_E(\alpha)$ (*cf.* 2.7 Proposition). Thus E is contained in a parameter field P/F for θ. We let P_m/P be unramified of degree m, with $P_m^\times \subset \boldsymbol{J}_\theta$. Let K_m/F be the maximal unramified sub-extension of P_m/F.

Let $\lambda_\tau \in \mathcal{T}_m(\mathbf{0}_E)$ be the extended maximal simple type occurring in ${}^L\tau$: this has been described in 7.5. Using this notation, we describe a maximal simple type occurring in Σ.

Types Theorem.

(1) *There exists $\nu \in \mathcal{H}(\theta)$ such that $\operatorname{tr}\nu$ is constant on the set of K_m/F-regular elements of $\boldsymbol{\mu}_{K_m}$. This condition determines $\nu|_{J^0_\theta}$ uniquely.*

(2) *The representation ${}^L\Sigma$ contains an element Λ of $\mathcal{T}(\theta)$ of the form*

$$\Lambda = \psi \odot \lambda_\tau \ltimes \nu,$$

where $\psi \in X_1(E)$ satisfies the following condition. If $e(E|F)$ is odd, then ψ is unramified, while $\psi|_{U_E}$ has order 2 if $e(E|F)$ is even.

We prove this result in §12.

7.7. We describe briefly a variant on the Types Theorem. In the same situation as 7.6, let $\sigma_1 = \mathrm{Ind}_{E/F}\,\rho \in \mathcal{G}_1^0(F;\Theta)$, and set $\pi_1 = {}^L\sigma_1$. In particular, $\dim \sigma_1 = n_1 = n/m$. Let $\vartheta = {}^L\tau \in \mathcal{A}_m^0(E;\mathbf{0}_E)$.

We choose an m-simple character θ_1 in $\mathrm{GL}_{n_1}(F)$ of endo-class Θ. The representation π_1 then contains an element κ_1 of $\mathcal{H}(\theta_1) = \mathcal{T}(\theta_1)$, while ϑ contains some $\lambda \in \mathcal{T}_m^0(\mathbf{0}_E)$. Using the notation of 3.3 Corollary 1, the representation κ_1 gives rise to a representation $\kappa_m = f_m(\kappa_1) \in \mathcal{H}(\theta)$. We set $\Lambda' = \lambda \ltimes \kappa_m \in \mathcal{T}(\theta)$ and form $\pi_m = c\text{-}\mathrm{Ind}_{\boldsymbol{J}_\theta}^G\,\Lambda' \in \mathcal{A}_m^0(F;\Theta)$, where $G = \mathrm{GL}_n(F)$. One sees easily that the representation π_m is of the form

$$\pi_m = \phi \odot_{\Phi_E(\alpha)} {}^L\Sigma, \tag{7.7.1}$$

for some $\phi \in X_1(E)$ depending only on m and α.

7.8. The first step (7.1.1) in the definition of the naïve correspondence is surely the natural one. However, any family of maps satisfying (7.1.2) and (7.1.3) would lead to the same outcome: only the character $\mu_{m,\alpha}^F$ would change by a constant factor depending on this choice.

More particularly, in the context of 7.2, each of $\mathcal{G}_1^0(E_m;\alpha)$, $\mathcal{A}_1^0(E_m;\Phi_{E_m}(\alpha))$ carries an action of the group $\Delta \ltimes X_1(E_m)$, and each is $\Delta \ltimes X_1(E_m)$-isomorphic to $X_1(E_m)$. Any two $\Delta \ltimes X_1(E_m)$-bijections of $\mathcal{G}_1^0(E_m;\alpha)$ with $\mathcal{A}_1^0(E_m;\Phi_{E_m}(\alpha))$ therefore differ by a constant element of $X_1(E_m)^\Delta$.

There is indeed a plausible alternative to (7.1.1), better preserving the spirit of explicitness. Let $\rho \in \widehat{\mathcal{W}}_F$ be totally wildly ramified of dimension p^r, $r \geqslant 1$. In [4], using methods not dissimilar to those of this paper, we constructed a totally ramified cuspidal representation ${}^W\!\rho$ of $\mathrm{GL}_{p^r}(F)$. The "totally wild" correspondence $\rho \mapsto {}^W\!\rho$ has properties (7.1.2) and (7.1.3). There is an unramified

character χ_α of $F^\times$, of order dividing p^{r-1} and depending only on $\alpha = \rho|_{\mathcal{P}_F}$, such that ${}^W\!\rho = \chi_\alpha \cdot {}^L\!\rho$. In the definition of the naïve correspondence $\sigma \mapsto {}^N\!\sigma$, we could equally well start by setting ${}^N\!\sigma = {}^W\!\sigma$ when σ is totally wildly ramified. The relationship between the Langlands correspondence and the totally wild correspondence is discussed in [5], [6].

8. Totally ramified representations

We prove the Comparison Theorem for *totally ramified* representations. Thus, in terms of the statement in 7.3, we have $m = 1$ and E/F is totally tamely ramified of degree e. We consequently revert to the notation listed at the beginning of 5.3.

8.1. We work relative to a simple character $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ in $G = \mathrm{GL}_n(F)$ such that $\Theta = cl(\theta) = \Phi_F(\alpha)$ and $\deg \Theta = n$. Thus $P = F[\beta]/F$ is totally ramified of degree n. We identify E/F with the maximal tamely ramified sub-extension of P/F, in such a way that $\Theta_E = cl(\theta_E) = \Phi_E(\alpha)$ (*cf.* 2.7). We use the standard abbreviations $\boldsymbol{J} = \boldsymbol{J}(\beta, \mathfrak{a}) = \boldsymbol{J}_\theta$ and so on. We follow the notational conventions, relative to centralizers of subfields of $\mathrm{M}_n(F)$, laid out in 2.3.

Set $\Gamma = \mathrm{Aut}(E|F)$. Let N_E denote the G-normalizer $N_G(E^\times)$ of $E^\times$. If $\gamma \in \Gamma$, there exists $g_\gamma \in N_E$ such that $g_\gamma^{-1} x g_\gamma = x^\gamma$, for every $x \in E$. The element g_γ is uniquely determined modulo G_E, and $g_\gamma \mapsto \gamma$ provides a canonical isomorphism $N_E/G_E \cong \Gamma$. If τ is a smooth representation of G_E, the equivalence class of the representation $\tau^\gamma : x \mapsto \tau(g_\gamma x g_\gamma^{-1})$ then depends only on that of τ, and not on the choice of g_γ.

We define the constant $\epsilon = \epsilon_{E/F} = \pm 1$ by (5.3.1) (*cf.* (5.3.2)). For $\pi \in \mathcal{A}_1^0(F; \Theta)$, we define $\pi_E \in \mathcal{A}_1^0(E; \Theta_E)$ as in 5.3 Corollary 1. As usual, we denote by $G_{\mathrm{reg}}^{\mathrm{ell}}$ the set of *elliptic regular* elements of G.

Proposition. *Let $\pi \in \mathcal{A}_1^0(F; \Theta)$. If $h \in G_E \cap G_{\mathrm{reg}}^{\mathrm{ell}}$ and $\upsilon_F(\det h)$ is relatively prime to n, then*

(8.1.1) $$\mathrm{tr}\, \pi(h) = \epsilon_{E/F} \sum_{\gamma \in \Gamma} \mathrm{tr}\, \pi_E^\gamma(h).$$

Proof. We write $\pi = c\text{-}\mathrm{Ind}_{\boldsymbol{J}}^G \Lambda$, where $\Lambda \in \mathcal{T}(\theta)$. Thus $\pi_E = c\text{-}\mathrm{Ind}_{\boldsymbol{J}_E}^{G_E} \Lambda_E$, where $\Lambda_E \in \mathcal{T}(\theta_E)$ is given by (5.3.3).

We examine more closely the elements h of the statement.

Lemma 1. *Let $h \in G_E \cap G^{\mathrm{ell}}_{\mathrm{reg}}$ and assume that $\upsilon_F(\det h)$ is relatively prime to n.*

(1) *The algebra $F[h]$ is a field, and the extension $F[h]/F$ is totally ramified of degree n.*
(2) *The field $F[h]$ contains E, and E/F is the maximal tamely ramified sub-extension of $F[h]/F$.*

Proof. Since $h \in G^{\mathrm{ell}}_{\mathrm{reg}}$, the algebra $F[h]$ is a field, of degree n over F. The condition on $\upsilon_F(\det h)$ ensures that $F[h]/F$ is totally ramified. Since $F[h]$ is a maximal subfield of A, it is its own centralizer in A. Since h commutes with E, the field $F[h]$ must contain E. Therefore $[F[h]{:}E] = n/[E{:}F] = p^r$, whence (2) follows. $\square$

We henceforward view the character $x \mapsto \operatorname{tr}\Lambda(x)$ of Λ as a function on G, vanishing outside $\boldsymbol{J}$. For $g \in G^{\mathrm{ell}}_{\mathrm{reg}}$, the Mackey formula of [3] Appendix then gives

$$\operatorname{tr}\pi(g) = \sum_{x \in G/\boldsymbol{J}} \operatorname{tr}\Lambda(x^{-1}gx). \tag{8.1.2}$$

There are only finitely many non-zero terms in the right hand side of this expansion [14] 1.2 Lemma. Similar considerations apply to the representations π_E^γ of G_E. Since we will only be concerned with term-by-term comparisons of these character expansions, we may re-arrange terms at will.

We eliminate one class of elements h. Suppose that $h \in G_E \cap G^{\mathrm{ell}}_{\mathrm{reg}}$ has no G-conjugate lying in $\boldsymbol{J}$. It then follows from (8.1.2) that $\operatorname{tr}\pi(h) = 0$. The element h also has no N_E-conjugate in $\boldsymbol{J}_E$, whence both sides of (8.1.1) vanish. So, we need only consider those elements h of $G_E \cap G^{\mathrm{ell}}_{\mathrm{reg}}$ having a G-conjugate in $\boldsymbol{J}$.

Lemma 2.

(1) *Let $h \in G_E \cap G^{\mathrm{ell}}_{\mathrm{reg}}$, and suppose that $\upsilon_F(\det h)$ is relatively prime to n. Let $x \in G$, and suppose $x^{-1}hx \in \boldsymbol{J}$. There then exists $y \in N_E$ such that $x\boldsymbol{J} \cap N_E = y\boldsymbol{J}_E$.*

(2) *The map*

$$\begin{aligned} N_E/\boldsymbol{J}_E &\longrightarrow G/\boldsymbol{J}, \\ y\boldsymbol{J}_E &\longmapsto y\boldsymbol{J}, \end{aligned}$$

is injective.

Proof. In the present case, we have $\boldsymbol{J} = P^\times J^1$, whence $\boldsymbol{J} \cap N_E = \boldsymbol{J}_E$ (2.6 Proposition (2)) and (2) follows. In (1), let $h' = x^{-1}hx \in \boldsymbol{J}$, By the Conjugacy Lemma (2.6) and 5.5 Lemma, there exists $j \in J^1$ such that $h'' = j^{-1}h'j \in \boldsymbol{J}_E$. Surely $h'' \in G_E \cap G_{\mathrm{reg}}^{\mathrm{ell}}$, and we can apply Lemma 1: the field $F[h'']$ contains E and E/F is the maximal tamely ramified sub-extension of $F[h'']/F$. Conjugation by the element $y = xj$ gives an F-isomorphism $F[h] \to F[h'']$ which must carry E to itself. That is, $y \in N_E$ and so the lemma is proved. $\square$

We return to the expansion (8.1.2). Following Lemma 2, it reads

$$\operatorname{tr} \pi(h) = \sum_{x \in G/\boldsymbol{J}} \operatorname{tr} \Lambda(x^{-1}hx) = \sum_{y \in N_E/\boldsymbol{J}_E} \operatorname{tr} \Lambda(y^{-1}hy),$$

with $h \in G_E \cap G_{\mathrm{reg}}^{\mathrm{ell}}$. For $y \in N_E$, we have $y^{-1}hy \in \boldsymbol{J}$ if and only if $y^{-1}hy \in \boldsymbol{J}_E$. In all cases, therefore, (5.3.3) yields

$$\operatorname{tr} \Lambda(y^{-1}hy) = \epsilon_{E/F} \operatorname{tr} \Lambda_E(y^{-1}hy).$$

Consequently,

$$\begin{aligned} \operatorname{tr} \pi(h) &= \epsilon_{E/F} \sum_{y \in N_E/\boldsymbol{J}_E} \operatorname{tr} \Lambda_E(y^{-1}hy) \\ &= \epsilon_{E/F} \sum_{\gamma \in \Gamma} \sum_{z \in G_E/\boldsymbol{J}_E} \operatorname{tr} \Lambda_E(z^{-1}h^\gamma z) \\ &= \epsilon_{E/F} \sum_{\gamma \in \Gamma} \operatorname{tr} \pi_E(h^\gamma), \end{aligned}$$

as required. $\square$

We record a technical consequence for later use.

Corollary. *Let a be an integer, relatively prime to n. There exists $h \in \boldsymbol{J}_E \cap G^{\mathrm{ell}}_{\mathrm{reg}}$ such that $\upsilon_F(\det h) = a$ and*

$$\operatorname{tr}\pi(h) = \epsilon_{E/F} \sum_{\gamma\in\Gamma} \operatorname{tr}\pi_E^\gamma(h) \neq 0.$$

Proof. The representation π is totally ramified, in that $t(\pi) = 1$ (*cf.* 6.3 Lemma 1). The representations $\chi\pi$, $\chi \in X_0(F)_n$, are therefore distinct and the character functions $\operatorname{tr}\chi\pi$ are linearly independent on $G^{\mathrm{ell}}_{\mathrm{reg}}$. It follows that there exists $h \in G^{\mathrm{ell}}_{\mathrm{reg}}$ such that $\upsilon_F(\det h) = a$ and $\operatorname{tr}\pi(h) \neq 0$. The Mackey formula (8.1.2) shows we may as well take $h \in \boldsymbol{J} = P^\times J^1$. Applying the Conjugacy Lemma of 2.6 and 5.5 Lemma, there exists $j \in J^1$ such that $h' = j^{-1}hj \in \boldsymbol{J}_E$. The element h' has all the desired properties. $\square$

8.2. Let $\rho \in \mathcal{G}^0_1(E;\alpha)$. We put

$$\nu_\rho = {}^L\rho, \quad \pi_\rho = {}^L(\operatorname{Ind}_{E/F}\rho). \tag{8.2.1}$$

Thus $\nu_\rho \in \mathcal{A}^0_1(E;\Theta_E)$ and $\pi_\rho \in \mathcal{A}^0_1(F;\Theta)$ (6.4 Corollary).

The representation $(\pi_\rho)_E$ lies in $\mathcal{A}^0_1(E;\Theta_E)$ (5.3 Corollary 1) and the set $\mathcal{A}^0_1(E;\Theta_E)$ is a principal homogeneous space over $X_1(E)$ (4.2 Proposition). It follows that there is a unique $\phi_\rho \in X_1(E)$ satisfying $(\pi_\rho)_E \cong \phi_\rho\nu_\rho$. We prove:

Proposition. *The function*

$$\rho \longmapsto \phi_\rho, \quad \rho \in \mathcal{G}^0_1(E;\alpha), \tag{8.2.2}$$

is constant.

Before starting the proof of the proposition, we note that it implies the Comparison Theorem in this case:

Corollary. *There is a unique character $\mu = \mu^F_{1,\alpha} \in X_1(E)$ such that*

$$\begin{aligned} {}^L(\operatorname{Ind}_{E/F}\sigma) &= \mathit{ind}_{E/F}\,\mu\cdot{}^L\sigma \\ &= \mu \odot_{\Theta_E} {}^N\sigma, \quad \sigma \in \mathcal{G}^0_1(E;\alpha). \end{aligned} \tag{8.2.3}$$

Proof. Continuing with the notation of the proposition, let ϕ denote the character ϕ_ρ, defined by the condition $(\pi_\rho)_E = \phi\nu_\rho$, for all $\rho \in \mathcal{G}^0_1(E;\alpha)$.

Let $\xi \in X_1(E)$ and consider the representation $\pi_{\xi\otimes\rho}$. According to the proposition,

$$(\pi_{\xi\otimes\rho})_E = \phi\nu_{\xi\otimes\rho} = \phi\xi\,\nu_\rho.$$

Invoking 5.3 Corollary 1, this relation is equivalent to $\pi_{\xi\otimes\rho} = \xi \odot_{\Theta_E} \pi_\rho$. In other words, the map $\mathcal{G}_1^0(F;\mathcal{O}_F(\alpha)) \to \mathcal{A}_1^0(F;\Theta)$, induced by the Langlands correspondence, is an isomorphism of $X_1(E)$-spaces. However, the composite map

$$\mathcal{G}_1^0(F;\mathcal{O}_F(\alpha)) \longrightarrow \mathcal{G}_1^0(E;\alpha) \longrightarrow \mathcal{A}_1^0(E;\Theta_E) \longrightarrow \mathcal{A}_1^0(F;\mathcal{O}_F(\alpha))$$

is also an isomorphism of $X_1(E)$ spaces. Since $\mathcal{G}_1^0(F;\mathcal{O}_F(\alpha))$, $\mathcal{A}_1^0(F;\Theta)$ are principal homogeneous spaces over $X_1(E)$ (1.5 Corollary, 4.2 Proposition), these two maps differ by a constant translation. This is precisely the assertion of the corollary. $\square$

8.3. We prove 8.2 Proposition by induction on $n = [E{:}F]$.

We observe that $|\Gamma| = \gcd(e, q{-}1)$ where, we recall, $q = |\Bbbk_F|$ and $e = [E{:}F]$. We first treat the case where $|\Gamma| = 1$. Here, the norm map $\mathrm{N}_{E/F} : E^\times \to F^\times$ is surjective, and induces an isomorphism $E^\times/U_E^1 \cong F^\times/U_F^1$. It therefore induces an isomorphism $X_1(F) \to X_1(E)$, denoted $\chi \mapsto \chi_E$. Let $\sigma \in \mathcal{G}_1^0(F;\mathcal{O}_F(\alpha))$, $\phi \in X_1(E)$. Writing $\phi = \chi_E$, $\chi \in X_1(F)$, we get

$${}^L(\phi \odot_\alpha \sigma) = {}^L(\chi \otimes \sigma) = \chi \cdot {}^L\sigma = \phi \odot_{\Theta_E} {}^L\sigma.$$

The Langlands Correspondence thus induces an isomorphism $\mathcal{G}_1^0(F;\mathcal{O}_F(\alpha)) \to \mathcal{A}_1^0(F;\Theta)$ of principal homogeneous spaces over $X_1(E)$. The same applies to the naïve correspondence. The two correspondences therefore differ by a constant translation. The Comparison Theorem follows in this case, and with it the proposition of 8.2.

Remark. In this case, the ramification index $e = e(E|F)$ is odd, so the discriminant character $d_{E/F}$ is unramified, of order $\leqslant 2$. It follows easily that the character $\mu = \mu_{1,\alpha}^F$ of the Comparison Theorem is unramified, of order dividing $2n$.

8.4. We therefore assume that $|\Gamma| \neq 1$, and we let l be the largest prime divisor of $|\Gamma|$. Let K/F be the unique sub-extension of E/F of degree l. Thus K/F is cyclic. We set $\Omega = \mathrm{Gal}(K/F)$.

Let $\rho \in \mathcal{G}_1^0(E;\alpha)$ and define $\tau_\rho = {}^L(\mathrm{Ind}_{E/K}\,\rho) \in \mathcal{A}_{n/l}^0(K)$. Taking $\pi_\rho = {}^L(\mathrm{Ind}_{E/F}\,\rho) \in \mathcal{A}_n^0(F)$ as in 8.2, the representation π_ρ is *automorphically induced* by τ_ρ. We use the notation $\pi_\rho = \mathrm{A}_{K/F}\,\tau_\rho$.

The representation π_ρ lies in $\mathcal{A}_1^0(F;\Theta)$, so $\pi_\rho \cong c\text{-}\mathrm{Ind}_{\boldsymbol{J}}^G\,\Lambda$, for a unique element Λ of $\mathcal{T}(\theta)$. In particular, π_ρ contains the simple character $\theta \in \mathcal{C}(\mathfrak{a},\beta)$ of 8.1, and E is identified with a subfield of $P = F[\beta]$.

Following the conventions of [13] §1, we choose a *transfer system* for K/F, in relative dimension n/l and based on a character $\varkappa$ of $F^\times$. This gives us a transfer factor $\boldsymbol{\delta} = \boldsymbol{\delta}_{K/F}$. By definition, the kernel of $\varkappa$ is the norm group $\mathrm{N}_{K/F}(K^\times)$. Therefore $\varkappa$ is trivial on $\det \boldsymbol{J} \subset \mathrm{N}_{E/F}(E^\times)U_F^1$, and we may apply the Uniform Induction Theorem of [13] 1.3. This gives the relation

$$\mathrm{tr}^\varkappa\,\pi_\rho(h) = c_\alpha^{K/F}\,\boldsymbol{\delta}(h)\sum_{\omega\in\Omega}\mathrm{tr}\,\tau_\rho^\omega(h), \tag{8.4.1}$$

valid for all $h \in G_K \cap G_{\mathrm{reg}}^{\mathrm{ell}}$. In (8.4.1), the constant $c_\alpha^{K/F}$ depends on the choice of transfer system (which we regard as fixed for all time). Otherwise, it depends only on the endo-class $cl(\theta_K) = \Phi_K(\alpha) = \mathfrak{i}_{E/K}\Phi_E(\alpha)$, and so only on α. The transfer factor $\boldsymbol{\delta}$ is independent of all considerations of representations. The term $\mathrm{tr}^\varkappa\,\pi_\rho$ is the normalized $\varkappa$-twisted trace of π_ρ, formed relative to the θ-normalized $\varkappa$-operator $\Phi_{\pi_\rho}^\varkappa$ on π_ρ (as in [13] 1.3). It satisfies a Mackey formula

$$\mathrm{tr}^\varkappa\,\pi_\rho(h) = \sum_{x\in G/\boldsymbol{J}_F}\varkappa(\det x^{-1})\,\mathrm{tr}\,\Lambda(x^{-1}hx), \quad h \in G_K \cap G_{\mathrm{reg}}^{\mathrm{ell}}, \tag{8.4.2}$$

and this is the only property we shall use.

We recall that if we treat τ_ρ as fixed, then the relation (8.4.1) determines π_ρ uniquely. Conversely, if we view π_ρ as given, then (8.4.1) determines the orbit $\{\tau_\rho^\omega : \omega \in \Omega\}$.

Let $h \in G_E \cap G_{\mathrm{reg}}^{\mathrm{ell}}$, and suppose that $\upsilon_F(\det h)$ is relatively prime to n. Using (8.4.2), the argument of 8.1 Proposition applies unchanged to give

$$\mathrm{tr}^\varkappa\,\pi_\rho(h) = \epsilon_{E/F}\sum_{\gamma\in\Gamma}\varkappa(\det g_\gamma^{-1})\,\mathrm{tr}\,\pi_{\rho,E}^\gamma(h). \tag{8.4.3}$$

Lemma. *Let $h \in G_E \cap G_{\mathrm{reg}}^{\mathrm{ell}}$ and $\omega \in \Omega$. Suppose that $\upsilon_F(\det h)$ is relatively prime to n and $\operatorname{tr}\tau_\rho^\omega(h) \neq 0$. The automorphism ω then extends to an F-automorphism of E.*

Proof. Choose $f_\omega \in N_K = N_G(K^\times)$ such that $f_\omega^{-1} z f_\omega = z^\omega$, for all $z \in K^\times$. Since $\operatorname{tr}\tau_\rho^\omega(h) = \operatorname{tr}\tau_\rho(f_\omega h f_\omega^{-1}) \neq 0$, the Mackey formula (8.1.2) implies that $h' = f_\omega h f_\omega^{-1}$ has a G_K-conjugate $h'' = x^{-1}h'x$ lying in $\boldsymbol{J}_K$. By the Conjugacy Lemma 2.6, there exists $j \in J_K^1$ such that $h_1 = j^{-1}h''j \in \boldsymbol{J}_E$. The field $K[h_1]$ therefore contains E, and E/K is the maximal tamely ramified sub-extension of $K[h_1]/K$. Likewise, E/K is the maximal tamely ramified sub-extension of $K[h]/K$. Conjugation by the element $y = f_\omega^{-1}xj$ thus induces an F-isomorphism $K[h] \to K[h_1]$, stabilizing E and extending the automorphism ω of K. Conjugation by y thus induces an automorphism of E/F with the desired property. $\square$

8.5. Before starting our analysis of the induction relation (8.4.1), we control the transfer factor $\boldsymbol{\delta}(h)$ for certain elements h. We use the notational conventions $\widetilde{\Delta}$, Δ^j, etc., of [13] (1.1.2).

Transfer Lemma. *Let $h_0 \in \boldsymbol{J}_E$, and suppose that $\upsilon_F(\det h_0)$ is relatively prime to n. The function $u \mapsto \boldsymbol{\delta}(h_0u)$, $u \in J_E^1$, is constant.*

Proof. There is an element x_0 of K such that $h_0^{n/l} \equiv x_0 \pmod{J_E^1}$. For any $h \in h_0J_E^1$, we also have $h^{n/l} \equiv x_0 \pmod{J_E^1}$. Let L/F be a finite Galois extension, containing K, all F-conjugates of h, n distinct n-th roots of unity and an (n/l)-th root x of x_0. An F-conjugate of h is of the form $\zeta x u$, for some $\zeta \in \boldsymbol{\mu}_L$ and some $u \in U_L^1$. The quantity $\widetilde{\Delta}(h)$ is therefore a product of terms $(\zeta - \zeta')$, for various $\zeta \neq \zeta' \in \boldsymbol{\mu}_L$, a power of x and a 1-unit of L. The differences of roots of unity and the element x depend only on the coset $h_0J_E^1$. Thus Δ^1 and Δ^2 are constant on this coset, so the same applies to $\boldsymbol{\delta} = \Delta^2/\Delta^1$. $\square$

8.6. It is now convenient to split into cases. For $\gamma \in \Gamma$, the element g_γ is determined modulo G_E, and G_E is contained in the kernel of the character $\varkappa \circ \det$ of G. Thus $\gamma \mapsto \varkappa(\det g_\gamma)$ is a character of Γ.

Lemma. *The character $\gamma \mapsto \varkappa(\det g_\gamma)$ of Γ is trivial unless the following condition is satisfied:*

$$l = |\Gamma| = 2 \quad \text{and} \quad e/2 \text{ is odd.} \tag{8.6.1}$$

Suppose condition (8.6.1) *holds (and so* $p \neq 2$*). The character is then trivial if* $q \equiv 1 \pmod 4$*, of order* 2 *if* $q \equiv 3 \pmod 4$.

Proof. By the Normal Basis Theorem, the automorphism g_γ of the E^Γ-vector space E is a permutation matrix. Its E^Γ-determinant is the signature of this permutation. Since $\varkappa$ has order l, the lemma follows immediately. $\square$

8.7. In this sub-section, we assume that condition (8.6.1) *fails.* Consequently, $\varkappa(\det g_\gamma) = 1$ for all $\gamma \in \Gamma$, and (8.4.3) is reduced to

$$\operatorname{tr}^\varkappa \pi_\rho(h) = \epsilon_{E/F} \sum_{\gamma\in\Gamma} \operatorname{tr} \pi^\gamma_{\rho,E}(h). \tag{8.7.1}$$

We deduce from (8.4.1) that

$$\epsilon_{E/F} \sum_{\gamma\in\Gamma} \operatorname{tr} \pi^\gamma_{\rho,E}(h) = c^{K/F}_\alpha\, \boldsymbol{\delta}(h) \sum_{\omega\in\Omega} \operatorname{tr} \tau^\omega_\rho(h). \tag{8.7.2}$$

We apply 8.4 Lemma to simplify the relation (8.7.2). Writing $\Delta = \operatorname{Aut}(E|K)$, we have an exact sequence

$$1 \to \Delta \longrightarrow \Gamma \longrightarrow \Omega.$$

Let Ω_0 denote the image of Γ in Ω: thus either $\Omega_0 = \Omega$ or Ω_0 is trivial. Either way, 8.4 Lemma implies

$$\epsilon_{E/F} \sum_{\gamma\in\Gamma} \operatorname{tr} \pi^\gamma_{\rho,E}(h) = c^{K/F}_\alpha\, \boldsymbol{\delta}(h) \sum_{\omega\in\Omega_0} \operatorname{tr} \tau^\omega_\rho(h).$$

Applying 8.1 Proposition, we expand the right hand side to get

$$\begin{aligned}\epsilon_{E/F} \sum_{\gamma\in\Gamma} \operatorname{tr} \pi^\gamma_{\rho,E}(h) &= \epsilon_{E/K}\, c^{K/F}_\alpha\, \boldsymbol{\delta}(h) \sum_{\omega\in\Omega_0} \sum_{\delta\in\Delta} \operatorname{tr} \tau^{\delta\omega}_{\rho,E}(h) \\ &= \epsilon_{E/K}\, c^{K/F}_\alpha\, \boldsymbol{\delta}(h) \sum_{\gamma\in\Gamma} \operatorname{tr} \tau^\gamma_{\rho,E}(h),\end{aligned} \tag{8.7.3}$$

the factor $\epsilon_{E/K}$ being a constant sign, given by (5.3.2), (5.3.3) relative to the base field K in place of F. It depends only on the simple stratum $[\mathfrak{a}_K, \beta]$.

As in (8.2.1), (8.2.2), ν_ρ denotes ${}^L\rho$ and $\pi_{\rho,E} = \phi_\rho \nu_\rho$, for some $\phi_\rho \in X_1(E)$. We have to show that ϕ_ρ is independent of the choice of $\rho \in \mathcal{G}^0_1(E;\alpha)$. By

inductive hypothesis, there is a character $\mu^K = \mu^K_{1,\alpha} \in X_1(E)$ such that $(\tau_\rho)_E = \mu^K \nu_\rho$, for all $\rho \in \mathcal{G}^0_1(E;\alpha)$.

We compare the central characters ω_{π_ρ}, ω_{τ_ρ} on $F^\times$. In (8.7.3), we replace h by zh, with $z \in F^\times$. The left hand side changes by a factor $\omega_{\pi_\rho}(z)$ and the right hand side by $\omega_{\tau_\rho}(z)\boldsymbol{\delta}(zh)/\boldsymbol{\delta}(h)$. Since $\pi_\rho = \mathrm{A}_{K/F}\,\tau_\rho$, the quotient $\omega^{-1}_{\pi_\rho}\omega_{\tau_\rho}|_{F^\times}$ is $d^{n/l}_{K/F}$, where $d_{K/F}$ is the determinant of the regular representation $\mathrm{Ind}_{K/F}\,1_K$ of $\mathcal{W}_K\backslash\mathcal{W}_F$. It is therefore unramified except when $l = 2$ and n/l is odd. Since l is the largest prime divisor of $|\Gamma|$, this case is excluded by 8.6 Lemma and the initial hypothesis of the sub-section.

So, $\boldsymbol{\delta}(zh)/\boldsymbol{\delta}(h)$ depends only on $\upsilon_F(z)$. The characters ϕ^γ_ρ, $(\mu^K)^\gamma$, thus all agree on $\boldsymbol{\mu}_F = \boldsymbol{\mu}_E$. We deduce that $\phi_\rho = \chi_\rho\mu^K$, where χ_ρ is unramified and, therefore, fixed by Γ. Thus (8.7.3) reduces to

$$\epsilon_{E/F}\,\chi_\rho(\det_E h)\sum_{\gamma\in\Gamma}\mathrm{tr}\,\tau^\gamma_{\rho,E}(h) = \epsilon_{E/K}\,c^{K/F}_\alpha\,\boldsymbol{\delta}(h)\sum_{\gamma\in\Gamma}\mathrm{tr}\,\tau^\gamma_{\rho,E}(h). \tag{8.7.4}$$

Let a be an integer relatively prime to n. By 8.1 Corollary, we may choose $h \in \boldsymbol{J}_E \cap G^{\mathrm{ell}}_{\mathrm{reg}}$ with $\upsilon_F(\det h) = \upsilon_E(\det_E h) = a$, such that $\sum_\gamma \mathrm{tr}\,\tau^\gamma_{\rho,E}(h) \neq 0$. We deduce that

$$\chi_\rho(\det_E h) = \epsilon_{E/F}\,\epsilon_{E/K}\,c^{K/F}_\alpha\,\boldsymbol{\delta}(h), \tag{8.7.5}$$

for all such $h \in \boldsymbol{J}_E \cap G^{\mathrm{ell}}_{\mathrm{reg}}$. The Transfer Lemma of 8.5 implies that (8.7.5) holds for all $h \in \boldsymbol{J}_E$ with $\upsilon_E(\det_E h)$ relatively prime to n. It follows that χ_ρ is independent of ρ, as desired.

8.8. We are reduced to the case where (8.6.1) *holds,* that is, where $|\Gamma| = 2$ and $[E^\Gamma{:}F]$ is odd. The canonical map $\Gamma \to \Omega = \mathrm{Gal}(K/F)$ is therefore an isomorphism. Let γ be the non-trivial element of Γ.

The analogue of (8.7.2) reads

$$\begin{aligned}\epsilon_{E/F}\,&\big(\mathrm{tr}\,\pi_{\rho,E}(h) + \varkappa(-1)\mathrm{tr}\,\pi^\gamma_{\rho,E}(h)\big)\\ &= \epsilon_{E/K}\,c^{K/F}_\alpha\,\boldsymbol{\delta}(h)\big(\mathrm{tr}\,\tau_{\rho,E}(h) + \mathrm{tr}\,\tau^\gamma_{\rho,E}(h)\big).\end{aligned} \tag{8.8.1}$$

As before, set $\nu_\rho = {}^L\rho$. There is a character $\phi_\rho \in X_1(E)$ such that $\pi_{\rho,E} = \phi_\rho\nu_\rho$. By inductive hypothesis, there is a character $\mu^K = \mu^K_{1,\alpha} \in X_1(E)$ such that $\tau_{\rho,E} = \mu^K\nu_\rho$ for all choices of $\rho \in \mathcal{A}^0_1(E;\alpha)$. We set $\chi_\rho = \phi_\rho/\mu^K$.

Comparing the central characters of π_ρ and τ_ρ on $F^\times$, we find:

Lemma 1.

(1) *If* $z \in F^\times$, *then* $\boldsymbol{\delta}(zh)/\boldsymbol{\delta}(h) = \varkappa(z)$ *and*
(2) $\chi_\rho|_{\boldsymbol{\mu}_F} = \varkappa|_{\boldsymbol{\mu}_F}$, *this character having order* 2.

We now invoke an elementary point.

Lemma 2. *Let* $\xi \in X_1(E)$ *and* $y \in E^\times$. *If* $\xi|_{\boldsymbol{\mu}_F}$ *has order* 2 *and* $\upsilon_E(y)$ *is odd, then* $\xi^\gamma(y) = \varkappa(-1)\xi(y)$.

Taking the lemmas into account, the relation (8.8.1) reads

$$\begin{aligned}\epsilon_{E/F}\,\chi_\rho(\det_E h)\big(\operatorname{tr}\tau_{\rho,E}(h) + \operatorname{tr}\tau^\gamma_{\rho,E}(h)\big)&\\ = c_\alpha^{K/F}\,\boldsymbol{\delta}(h)\big(\operatorname{tr}\tau_{\rho,E}(h) + \operatorname{tr}\tau^\gamma_{\rho,E}(h)\big)&.\end{aligned}$$

That is,

$$\chi_\rho(\det_E h) = \epsilon_{E/K}\epsilon_{E/F}\,c_\alpha^{K/F}\,\boldsymbol{\delta}(h), \tag{8.8.2}$$

for all $h \in \boldsymbol{J}_E \cap G^{\rm ell}_{\rm reg}$ with $\upsilon_E(\det_E h)$ odd and $\operatorname{tr}\tau_{\rho,E}(h) + \operatorname{tr}\tau^\gamma_{\rho,E}(h) \neq 0$. Arguing as before, the character χ_ρ is independent of ρ.

This completes the proof of 8.2 Proposition, and hence that of the Comparison Theorem in the present case. $\square$

8.9. For ease of reference, we display an intermediate conclusion of the preceding argument. We use the notation of the Comparison Theorem in 7.3.

Corollary. *Suppose that* E/F *is totally ramified and that* $\Gamma = \operatorname{Aut}(E|F)$ *is non-trivial. Let* l *be the largest prime divisor of* $|\Gamma|$ *and let* K/F *be the unique sub-extension of* E/F *of order* l. *There is a unique character* $\chi^K \in X_1(E)$ *such that*

$$\chi^K(\det_E h) = \epsilon_{E/F}\epsilon_{E/K}\,c_\alpha^{K/F}\,\boldsymbol{\delta}_{K/F}(h), \tag{8.9.1}$$

for all $h \in \boldsymbol{J}_E$ *with* $\upsilon_E(\det_E h)$ *relatively prime to* n. *The character* $\mu^F_{1,\alpha}$ *then satisfies*

$$\mu^F_{1,\alpha} = \chi^K \cdot \mu^K_{1,\alpha}. \tag{8.9.2}$$

We recall that the factors $\epsilon_{E/F}$, $\epsilon_{E/K}$ are symplectic signs: for example, $\epsilon_{E/F}$ is $t_{E^\times/F^\times U^1_E}(J^1_\theta/H^1_\theta) = \pm 1$. In the essentially tame case, one may compute the character $\mu^F_{1,\alpha}$ iteratively, using (8.9.2). The first step, where $\Gamma = \mathrm{Aut}(E|F)$ is trivial, requires a base change argument as in [10] 4.6. The steps coming from automorphic induction are computed in [10] 4.5 and [11] *passim.*

We comment briefly on the general case. For the first step, as in 8.3, a base change argument analogous to that of [10] could not determine the desired character directly. In the cyclic case of the corollary, one may look at values $\chi^K(h_1h_2^{-1})$, for elements h_i of $\boldsymbol{J}_E$ with $\upsilon_E(\det_E h_i)$ relatively prime to n. Suppose, for a simple example, that l and p are odd. The discriminant character $d_{K/F}$ is then trivial, so $\chi^K\circ\det_E$ is then trivial on $F^\times$. Since p is odd, one may choose the h_i so that $\upsilon_E(\det_E h_1h_2^{-1})$ is divisible by e but relatively prime to p. The transfer factors $\boldsymbol{\delta}_{K/F}(h_i)$ are each the product of an l-th root of unity and a positive real number. One concludes that χ^K has order dividing l and that $\epsilon_F\epsilon_K c_\alpha > 0$. An argument similar to that of [10] 4.5 Lemma now implies that χ^K is trivial.

9. Unramified automorphic induction

In this section and the next, we cover the main part of the proof of the Comparison Theorem. We work in the general case, using the notation laid out at the start of 5.4. In particular, E_0/F is a tame parameter field for θ, E/E_0 is unramified of degree m with $\Delta = \mathrm{Gal}(E/E_0)$, and K/F is the maximal unramified sub-extension of E/F. In that situation, we set $\Theta = cl(\theta)$ and use the notation Θ_K etc., as in (5.8.1). As usual, we abbreviate $J^k = J^k_\theta$, $J^k_K = J^k_{\theta_K} = J^k \cap G_K$, and so on. In addition, we set $d = [K{:}F]$.

9.1. We recall a property of automorphic induction.

Proposition. *The operation of automorphic induction induces a bijection*

$$\mathrm{A}_{K/F} : \Delta\backslash\mathcal{A}^0_1(K;\Theta_K)^{\Delta\text{-reg}} \xrightarrow{\approx} \mathcal{A}^0_m(F;\Theta). \tag{9.1.1}$$

Proof. The endo-classes Θ^γ_K, $\gamma \in \Delta\backslash\Gamma$, are distinct, as follows from (2.3.5). Consequently, any representation $\tau \in \mathcal{A}^0_1(K;\Theta_K)^{\Delta\text{-reg}}$ is Γ-regular. The automorphically induced representation $\pi = \mathrm{A}_{K/F}\,\tau$ is therefore cuspidal. The endo-class $\vartheta(\pi)$ is $\mathbf{i}_{K/F}\,\Theta_K = \Theta$ [8] 7.1 Corollary, whence $\pi \in \mathcal{A}^0_m(F;\Theta)$.

Conversely, let $\pi \in \mathcal{A}_m^0(F;\Theta)$. The integer $t(\pi)$ is $d = [K{:}F]$ (6.3 Lemma 2). It follows that $\pi \cong \mathrm{A}_{K/F}\,\tau$, for some Γ-regular $\tau \in \mathcal{A}_{n/d}^0(K)$. The representation τ is determined by π, up to Γ-conjugation. The endo-class $\vartheta(\tau)$ is a K/F-lift of $\vartheta(\pi) = \Theta$. The set of K/F-lifts of Θ form a single Γ-orbit, so we may choose τ to satisfy $\vartheta(\tau) = \Theta_K$. Therefore $\tau \in \mathcal{A}_1^0(K;\Theta_K)$. The representation τ is surely Δ-regular, and π determines its Δ-orbit. $\square$

We compare the bijection (9.1.1) with the bijection (5.8.2) given by the algebraic induction map $ind_{K/F}$. As before, we write $\chi_E = \chi \circ \mathrm{N}_{E/E_0}$, $\chi \in X_1(E_0)$, and recall that the map $\chi \mapsto \chi_E$ induces an isomorphism $X_1(E_0)/X_0(E_0)_m \to X_1(E)^{\Delta}$.

Unramified Induction Theorem. *There is a unique character $\nu \in X_1(E)^{\Delta}$ such that*

$$\mathrm{A}_{K/F}\,\tau = ind_{K/F}\,\nu \odot_{\Theta_E} \tau, \tag{9.1.2}$$

for all $\tau \in \mathcal{A}_1^0(K;\Theta_K)^{\Delta\text{-reg}}$. In particular,

$$\mathrm{A}_{K/F}(\chi_E \odot_{\Theta_E} \tau) = \chi \odot_{\Theta_{E_0}} \mathrm{A}_{K/F}\,\tau, \tag{9.1.3}$$

for $\chi \in X_1(E_0)$ and $\tau \in \mathcal{A}_1^0(K;\Theta_K)^{\Delta\text{-reg}}$.

The proof will occupy us until the end of §11. Before starting it, we remark that the second assertion of the theorem follows from the first and (5.8.3). The character ν is of the form $(\nu_0)_E$, for some $\nu_0 \in X_1(E_0)$. In those terms, the relation (9.1.2) reads

$$\mathrm{A}_{K/F}\,\tau = \nu_0 \odot_{\Theta_{E_0}} ind_{K/F}\,\tau. \tag{9.1.4}$$

While (9.1.2) determines ν uniquely, the character ν_0 is only determined modulo $X_0(E_0)_m$.

9.2. We choose a prime element ϖ_F of F and a representation $\kappa \in \mathcal{H}(\theta)$ satisfying the conditions of 5.6 Lemma 2. We define $\kappa_K = \ell_{K/F}(\kappa) \in \mathcal{H}(\theta_K)$, as in 5.6 Proposition. If $\xi \in X_1(E)^{\Delta\text{-reg}}$, we set

$$\tau_\xi = c\text{-}\mathrm{Ind}_{\boldsymbol{J}_K}^{G_K}\,\xi \odot \kappa_K \in \mathcal{A}_1^0(K;\Theta_K)^{\Delta\text{-reg}}. \tag{9.2.1}$$

Lemma 1 of 5.7 implies that every element of $\mathcal{A}_1^0(K;\Theta_K)^{\Delta\text{-reg}}$ is of this form, for some $\xi \in X_1(E)^{\Delta\text{-reg}}$. Likewise, we define

$$\pi_\xi = c\text{-}\mathrm{Ind}_{\boldsymbol{J}}^{G}\,\lambda_\xi \ltimes \kappa = ind_{K/F}\,\tau_\xi \in \mathcal{A}_m^0(F;\Theta), \tag{9.2.2}$$

in the notation of (3.6.2).

Lemma. *There exists* $\chi = \chi_\xi \in X_1(E)^{\Delta\text{-reg}}$ *such that*

$$\mathrm{A}_{K/F}\,\tau_\xi = \pi_\chi. \tag{9.2.3}$$

The correspondence $\xi \mapsto \chi_\xi$ *induces a bijection of* $\Delta\backslash X_1(E)^{\Delta\text{-reg}}$ *with itself.*

Proof. This follows from the bijectivity of the map (9.1.1) and 5.8 Proposition. □

In these terms, the Unramified Induction Theorem of 9.1 is equivalent to the following.

Theorem. *There is a unique character* $\mu = \mu_\alpha^{K/F} \in X_1(E)^{\Delta}$ *such that*

$$\mathrm{A}_{K/F}\,\tau_\xi = \pi_{\mu\xi} = \mathit{ind}_{K/F}\,\mu \odot_{\Theta_E} \tau_\xi,$$

for all $\xi \in X_1(E)^{\Delta\text{-reg}}$.

To prove the theorem, we use the automorphic induction equation,

$$\mathrm{tr}^{\varkappa}\,\pi_\chi(g) = c_\alpha^{K/F}\,\boldsymbol{\delta}(g)\sum_{\gamma\in\Gamma}\mathrm{tr}\,\tau_\xi^\gamma(g), \quad g \in G_K \cap G_{\mathrm{reg}}^{\mathrm{ell}}, \tag{9.2.4}$$

as in §8. In this case, $\varkappa$ is an unramified character of $F^\times$ of order $d = [K{:}F]$. It is trivial on the group $\det \boldsymbol{J}$, so we may again apply the Uniform Induction Theorem of [13] 1.3. The $\varkappa$-twisted trace $\mathrm{tr}^{\varkappa}\,\pi_\chi$ is defined so as to satisfy the Mackey formula

$$\mathrm{tr}^{\varkappa}\,\pi(g) = \sum_{x\in G/\boldsymbol{J}} \varkappa(\det x^{-1})\,\mathrm{tr}\,\Lambda(x^{-1}gx), \quad g \in G_K \cap G_{\mathrm{reg}}^{\mathrm{ell}}, \tag{9.2.5}$$

where $\Lambda_\chi = \lambda_\chi^{\boldsymbol{J}} \otimes \kappa = \lambda_\chi \ltimes \kappa$. The constant $c_\alpha^{K/F}$ depends only on θ and K, and $\boldsymbol{\delta} = \boldsymbol{\delta}_{K/F}$ is a transfer factor. We recall that there are only finitely many terms in the expansion (9.2.5) ([14] 1.2).

9.3. We evaluate each side of the relation (9.2.4) at a special family of elements.

We use the groups ${}_p\boldsymbol{J}(\varpi_F) = {}_p\boldsymbol{J}$ and ${}_p\boldsymbol{J}_K$ defined in 5.5. Recalling that $p^r = [P{:}E]$, we say that $g \in$ is *pro-unipotent in* $G/\langle\varpi_F\rangle$ if $g^{p^{r+t}} \to 1$ in $G/\langle\varpi_F\rangle$ as $t \to \infty$. In particular, any element of ${}_p\boldsymbol{J}$ is pro-unipotent in $G/\langle\varpi_F\rangle$.

Lemma 1. *Let $g = \zeta h$, where $\zeta \in \boldsymbol{\mu}_K$ is Γ-regular and $h \in {}_p\boldsymbol{J}_K \cap (G_K)^{\mathrm{ell}}_{\mathrm{reg}}$. The element g then lies in $G_K \cap G^{\mathrm{ell}}_{\mathrm{reg}}$. Let $x \in G$ satisfy $x^{-1}gx \in \boldsymbol{J}$. The coset $x\boldsymbol{J}$ then contains an element y of $N_K = N_G(K^\times)$. For any such y, we have $h' = y^{-1}hy \in {}_p\boldsymbol{J}_K$.*

Proof. The first assertion is immediate.

Of the element $g = \zeta h$, the factors ζ, h commute with each other; the first has finite order prime to p, and the second is pro-unipotent in $G/\langle \varpi_F \rangle$. The factors ζ, h are thereby uniquely determined. So, any closed subgroup of G containing g and ϖ_F must contain both ζ and h. If $x \in G$ and if $g' = x^{-1}gx$ lies in $\boldsymbol{J}$, it follows that both $\zeta' = x^{-1}\zeta x$ and $h' = x^{-1}hx$ lie in $\boldsymbol{J}$. In particular, $\zeta' \in J^0$. The algebra $F[\zeta']$ is a field in which ζ' is a root of unity of the same order as ζ. It follows that ζ' is J^0-conjugate to an element of $\boldsymbol{\mu}_K$. Replacing x by $y = xj$, for suitable $j \in J^0$, we therefore get $g'' = y^{-1}gy \in \boldsymbol{J}$ and $\zeta'' = y^{-1}\zeta y \in \boldsymbol{\mu}_K$. Thus $y \in N_K$. Again, $h'' = y^{-1}hy$ lies in $\boldsymbol{J}$ and commutes with ζ''. It follows that $h'' \in \boldsymbol{J}_K = P^\times J^1_K$. Since h'' is pro-unipotent in $G/\langle \varpi_F \rangle$, it must lie in ${}_p\boldsymbol{J}_K$ as required. □

We extract a remark made in the course of the proof.

Lemma 2. *Let $h \in {}_p\boldsymbol{J}_K$ and $y \in G_K$. If the element $y^{-1}hy$ lies in $\boldsymbol{J}_K$ then it lies in ${}_p\boldsymbol{J}_K$.*

The natural action of N_K on $K^\times$ induces an isomorphism $N_K/G_K \to \Gamma$. For each $\gamma \in \Gamma$, we choose $g_\gamma \in N_K$ with image γ in Γ. The groups $N_K \cap \boldsymbol{J}$, $N_K \cap J^0$ both have image Δ in Γ. The map $x \mapsto x\boldsymbol{J}$ thus induces a bijection

$$\bigcup_{\gamma \in \Gamma/\Delta} g_\gamma G_K/\boldsymbol{J}_K \longrightarrow N_K\boldsymbol{J}/\boldsymbol{J},$$

the union being disjoint. The character $\varkappa \circ \det$ of G is trivial on G_K, while $\det g_\gamma \equiv \pm 1 \pmod{\det G_K}$ (as in the discussion of 8.6). Since $\varkappa$ is unramified, this implies $\varkappa(\det x) = 1$ for all $x \in N_K$. Thus (9.2.5) is reduced to

$$\operatorname{tr}^\varkappa \pi_\chi(\zeta h) = \sum_{\gamma \in \Gamma/\Delta} \sum_{y \in G_K/\boldsymbol{J}_K} \operatorname{tr} \Lambda_\chi(y^{-1}\zeta^\gamma h^\gamma y) = \operatorname{tr} \pi_\chi(\zeta h),$$

for any element $g = \zeta h$ as in Lemma 1.

We recall (9.2.2) that $\varLambda_\chi = \lambda_\chi^{\boldsymbol J}\otimes\kappa$. We evaluate the factor $\operatorname{tr}\kappa(y^{-1}\zeta^\gamma h^\gamma y)$, using (5.6.1). The character $\epsilon^1_{\boldsymbol\mu_K} = t^1_{\boldsymbol\mu_K}(J^1_\theta/H^1_\theta)$ of the cyclic group $\boldsymbol\mu_K$ has order $\leqslant 2$ and so is $\varGamma$-invariant. Therefore

$$\operatorname{tr}\kappa(y^{-1}\zeta^\gamma h^\gamma y) = \epsilon^0_{\boldsymbol\mu_K}\epsilon^1_{\boldsymbol\mu_K}(\zeta)\operatorname{tr}\kappa_K(y^{-1}h^\gamma y),$$

where $\kappa_K = \ell_{K/F}(\kappa)$ (as in the definition of τ_ξ) and $\epsilon^0_{\boldsymbol\mu_K} = t^0_{\boldsymbol\mu_K}(J^1_\theta/H^1_\theta) = \pm1$.

We consider the term $\operatorname{tr}\lambda_\chi^{\boldsymbol J}(\zeta^\gamma y^{-1}h^\gamma y)$. As in 3.1 (but with base field K), we have the character $\chi^{\boldsymbol J_K}\in X_1(\theta_K)$ satisfying

$$\chi^{\boldsymbol J_K}|_{P^\times} = \chi_P = \chi\circ \mathrm N_{P/E}.$$

The group ${}_p\boldsymbol J_K$ is contained in the subgroup $P_0^\times J^1_K$ of $\boldsymbol J_K = P^\times J^1_K$. The restriction of $\lambda_\chi^{\boldsymbol J}$ to $P_0^\times J^1_K$ is a multiple of the character $\chi^{\boldsymbol J_K}|_{P_0^\times J^1_K}$. Let $X_0(\theta_K)$ denote the subgroup of $X_1(\theta_K)$ consisting of characters trivial on $J^0_K = \boldsymbol\mu_K J^1_K$: thus $X_0(\theta_K)$ is the image of $X_0(E)$ under the map $\phi\mapsto\phi^{\boldsymbol J_K}$ of (3.1.1).

We choose a character $\tilde a_\chi\in X_0(\theta_K)$ to agree with $\chi^{\boldsymbol J_K}$ on ${}_p\boldsymbol J_K$. This is of the form $\tilde a_\chi = a_\chi^{\boldsymbol J_K}$, for a character $a_\chi\in X_0(E)$, uniquely determined modulo $X_0(E)_e$, $e = e(E|F) = [E{:}K]$. We get

$$\operatorname{tr}\lambda_\chi^{\boldsymbol J}(\zeta^\gamma y^{-1}h^\gamma y) = \operatorname{tr}\lambda_\chi^{\boldsymbol J}(\zeta^\gamma)\,\tilde a_\chi(y^{-1}h^\gamma y) = (-1)^{m-1}\tilde a_\chi(y^{-1}h^\gamma y)\sum_{\delta\in\varDelta}\chi_P(\zeta^{\gamma\delta}),$$

and so

$$\operatorname{tr}\varLambda_\chi(\zeta^\gamma y^{-1}h^\gamma y) = (-1)^{m-1}\epsilon^0_{\boldsymbol\mu_K}\epsilon^1_{\boldsymbol\mu_K}(\zeta)\operatorname{tr}(a_\chi\odot\kappa_K)(y^{-1}h^\gamma y)\sum_{\delta\in\varDelta}\chi_P(\zeta^{\gamma\delta}).$$

By definition, $\kappa_K\in\mathcal H(\theta_K)^\varDelta = \mathcal T(\theta_K)^\varDelta$, so

$$\rho = c\text{-}\mathrm{Ind}_{\boldsymbol J_K}^{G_K}\kappa_K \tag{9.3.1}$$

is an irreducible, cuspidal representation of G_K, lying in $\mathcal A^0_1(K;\varTheta_K)^\varDelta$. Assembling the preceding identities, and taking account of Lemma 2, we obtain

(9.3.2)

$$\begin{aligned}\operatorname{tr}^\varkappa\pi_\chi(\zeta h) &= (-1)^{m-1}\epsilon^0_{\boldsymbol\mu_K}\epsilon^1_{\boldsymbol\mu_K}(\zeta)\sum_{\gamma\in\varGamma/\varDelta}\Big(\sum_{\delta\in\varDelta}\chi_P(\zeta^{\gamma\delta})\Big)\operatorname{tr}(a_\chi\odot_{\varTheta_E}\rho)(h^\gamma)\\ &= (-1)^{m-1}\epsilon^0_{\boldsymbol\mu_K}\sum_{\gamma\in\varGamma/\varDelta}\Big(\sum_{\delta\in\varDelta}\epsilon^1_{\boldsymbol\mu_K}\chi_P(\zeta^{\gamma\delta})\Big)\operatorname{tr}(a_\chi\odot_{\varTheta_E}\rho)(h^\gamma).\end{aligned}$$

9.4. We look to the other side of the automorphic induction relation (9.2.4), using the same notation. We expand $\operatorname{tr}\tau_\xi(\zeta^\gamma h^\gamma)$. The representation κ_K is trivial on $\boldsymbol{\mu}_K$ (remark following 5.6 Lemma 3). From 9.3 Lemma 2, we obtain

$$\begin{aligned}\operatorname{tr}\tau_\xi(\zeta^\gamma h^\gamma) &= \xi_P(\zeta^\gamma)\sum_{y\in G_K/\boldsymbol{J}_K}\tilde a_\xi(y^{-1}h^\gamma y)\operatorname{tr}\kappa_K(y^{-1}h^\gamma y)\\ &= \xi_P(\zeta^\gamma)\operatorname{tr}(a_\xi\odot_{\Theta_E}\rho)(h^\gamma).\end{aligned}$$

Taking account of (9.3.2), the relation (9.2.4) now reads

$$\begin{aligned}(9.4.1)\quad (-1)^{m-1}\epsilon^0_{\boldsymbol{\mu}_K}&\sum_{\gamma\in\Gamma/\Delta}\Big(\sum_{\delta\in\Delta}\epsilon^1_{\boldsymbol{\mu}_K}\chi_P(\zeta^{\gamma\delta})\Big)\operatorname{tr}(a_\chi\odot_{\Theta_E}\rho)(h^\gamma)\\ &= c_\alpha^{K/F}\,\boldsymbol{\delta}(\zeta h)\sum_{\gamma\in\Gamma/\Delta}\Big(\sum_{\delta\in\Delta}\xi_P(\zeta^{\gamma\delta})\Big)\operatorname{tr}(a_\xi\odot_{\Theta_E}\rho)(h^\gamma).\end{aligned}$$

9.5. We simplify the relation (9.4.1), starting with a special case. Since $h\in {}_p\boldsymbol{J}_K$ and $\zeta\in\boldsymbol{\mu}_K$ is Γ-regular, the condition $\zeta h\in G^{\mathrm{ell}}_{\mathrm{reg}}$ is equivalent to $h\in(G_K)^{\mathrm{ell}}_{\mathrm{reg}}$.

Transfer Lemma. *If $h\in J^1_K\cap(G_K)^{\mathrm{ell}}_{\mathrm{reg}}$ and if $\zeta\in\boldsymbol{\mu}_K$ is Γ-regular, then $\boldsymbol{\delta}(\zeta h)=1$.*

Proof. Let $\beta,\gamma\in\Gamma$, $\beta\neq\gamma$. Let y_β, y_γ be eigenvalues of h^β, h^γ respectively, in some splitting field L/K. Thus $y_\beta,y_\gamma\in U^1_L$ and $\zeta^\beta y_\beta-\zeta^\gamma y_\gamma$ is a unit in L. We follow the standard notation laid out in [13] (1.1.2) and observe the convention of [13] Remark 1. In those terms, the quantity $\widetilde\Delta(\zeta h)$ is a unit and, since $\varkappa$ is unramified, the lemma follows. $\square$

The identity (9.4.1) now implies

$$\begin{aligned}(9.5.1)\quad (-1)^{m-1}\epsilon^0_{\boldsymbol{\mu}_K}&\sum_{\gamma\in\Gamma/\Delta}\Big(\sum_{\delta\in\Delta}\epsilon^1_{\boldsymbol{\mu}_K}\chi_P(\zeta^{\gamma\delta})\Big)\operatorname{tr}(a_\chi\odot_{\Theta_E}\rho)(h^\gamma)\\ &= c_\alpha^{K/F}\sum_{\gamma\in\Gamma/\Delta}\Big(\sum_{\delta\in\Delta}\xi_P(\zeta^{\gamma\delta})\Big)\operatorname{tr}(a_\xi\odot_{\Theta_E}\rho)(h^\gamma),\end{aligned}$$

for $h\in J^1_K\cap(G_K)^{\mathrm{ell}}_{\mathrm{reg}}$. The characters a_χ and a_ξ are unramified, so we get

$$\operatorname{tr}(a_\chi\odot_{\Theta_E}\rho)(h^\gamma)=\operatorname{tr}\rho(h^\gamma)=\operatorname{tr}(a_\xi\odot_{\Theta_E}\rho)(h^\gamma),$$

and the relation (9.5.1) becomes

$$(9.5.2)\quad (-1)^{m-1}\epsilon^0_{\boldsymbol{\mu}_K}\sum_{\gamma\in\Gamma/\Delta}\Big(\sum_{\delta\in\Delta}\epsilon^1_{\boldsymbol{\mu}_K}\chi_P(\zeta^{\gamma\delta})\Big)\operatorname{tr}\rho(h^\gamma) \\ = c^{K/F}_\alpha\sum_{\gamma\in\Gamma/\Delta}\Big(\sum_{\delta\in\Delta}\xi_P(\zeta^{\gamma\delta})\Big)\operatorname{tr}\rho(h^\gamma),\quad h\in J^1_K\cap(G_K)^{\mathrm{ell}}_{\mathrm{reg}}.$$

To proceed further, we need:

Linear Independence Lemma. *Let $h\in \boldsymbol{J}_K$, and let u range over the set of elements of J^1_K such that $hu\in(G_K)^{\mathrm{ell}}_{\mathrm{reg}}$. The set of functions*

$$u\longmapsto \operatorname{tr}\rho^\gamma(hu),\quad \gamma\in\Delta\backslash\Gamma,$$

is linearly independent.

We shall prove this lemma in 9.7 below. It implies

$$(-1)^{m-1}\epsilon^0_{\boldsymbol{\mu}_K}\sum_{\delta\in\Delta}\epsilon^1_{\boldsymbol{\mu}_K}\chi_P(\zeta^\delta) = c^{K/F}_\alpha\sum_{\delta\in\Delta}\xi_P(\zeta^\delta).$$

We recall that the constant $c^{K/F}_\alpha$ does not depend on ξ. The lemma of 9.2 therefore enables us to apply [13] 2.3 Corollary 1 and so obtain:

Proposition. *The characters ξ and $\epsilon^1_{\boldsymbol{\mu}_K}\chi$ of $\boldsymbol{\mu}_K$ lie in the same Δ-orbit, and*

$$c^{K/F}_\alpha = (-1)^{m-1}\,\epsilon^0_{\boldsymbol{\mu}_K}.$$

The first conclusion of the proposition may be strengthened.

Corollary. *Let $\xi\in X_1(E)^{\Delta\text{-reg}}$. There exists a unique character*

$$\chi=\chi_\xi\in X_1(E)^{\Delta\text{-reg}}$$

such that

$$\chi|_{U_E}=\epsilon^1_{\boldsymbol{\mu}_K}\xi|_{U_E}\quad\text{and}\quad \mathrm{A}_{K/F}\,\tau_\xi=\pi_\chi.$$

The character $\xi^{-1}\chi_\xi$ has finite order.

Proof. Given ξ, there surely exists $\chi \in X_1(E)^{\Delta\text{-reg}}$ satisfying the two requirements. The first of them determines χ modulo $X_0(E)$. We therefore take $\phi \in X_0(E)$, $\phi \neq 1$, and show that $\pi_{\phi\chi} \neq \pi_\chi$.

The character ϕ is of the form $\psi \circ \mathrm{N}_{E/F}$, for some $\psi \in X_0(F)$. We then have $\pi_{\phi\chi} = \psi\pi_\chi$. The relation $\psi\pi_\chi = \pi_\chi$ would imply $\psi \circ \mathrm{N}_{K/F} = 1$, which is false since $\psi \circ \mathrm{N}_{E/F} \neq 1$. Thus $\phi = 1$ as required.

For the final assertion, it is enough to show that $\chi^{-1}\xi(\varpi_F)$ has finite order. Let ψ_2 temporarily denote the unramified character of $F^\times$ of order 2. By construction, both κ and κ_K are trivial on ϖ_F. The central character ω_{π_χ} of π_χ thus satisfies $\omega_{\pi_\chi}(\varpi_F) = \chi_P(\varpi_F)$. On the other hand, the central character of $\mathrm{A}_{K/F}\,\tau_\xi$ is $\psi_2^{(n-d)}\,\omega_{\tau_\xi}|_{F^\times}$. Its value at ϖ_F is therefore $(-1)^{(n-d)}\xi_P(\varpi_F)$, whence $\chi^{-1}\xi(\varpi_F)^{p^r} = (-1)^{(n-d)}$. $\square$

In the situation of the corollary, we define

$$\mu_\xi = \xi^{-1}\chi_\xi, \quad \xi \in X^1(E)^{\Delta\text{-reg}}. \tag{9.5.3}$$

We show that μ_ξ does not depend on ξ: this will prove 9.2 Theorem, with $\mu_\alpha^{K/F}$ being the common value of the μ_ξ.

9.6. We return to the identity (9.4.1), relative to an arbitrary element h of ${}_p\boldsymbol{J}_K \cap (G_K)^{\mathrm{ell}}_{\mathrm{reg}}$. The character a_χ, by definition, lies in $X_0(E)$. So, there exists an unramified character b_χ of $K^\times$ such that $a_\chi = b_\chi \circ \mathrm{N}_{E/K}$. This gives

$$\mathrm{tr}(a_\chi \odot_{\Theta_E} \rho)(h^\gamma) = b_\chi(\det_K h^\gamma)\,\mathrm{tr}\,\rho(h^\gamma) = b_\chi(\det_K h)\,\mathrm{tr}\,\rho(h^\gamma).$$

Looking back at the definition of a_χ in 9.3, we see that $b_\chi(\det_K h) = \chi^{\boldsymbol{J}_K}(h)$. Similar considerations apply to the character a_ξ. Taking account of 9.5 Proposition, the relation (9.4.1) is therefore reduced to

$$\chi^{\boldsymbol{J}_K}(h) \sum_{\gamma\in\Gamma/\Delta} \mathrm{tr}\,\rho(h^\gamma) = \boldsymbol{\delta}(\zeta h)\,\xi^{\boldsymbol{J}_K}(h) \sum_{\gamma\in\Gamma/\Delta} \mathrm{tr}\,\rho(h^\gamma). \tag{9.6.1}$$

This relation is valid for all $h \in {}_p\boldsymbol{J}_K \cap (G_K)^{\mathrm{ell}}_{\mathrm{reg}}$. The characters χ, ξ therefore satisfy

$$\chi^{\boldsymbol{J}_K}(h) = \boldsymbol{\delta}(\zeta h)\,\xi^{\boldsymbol{J}_K}(h), \tag{9.6.2}$$

for any $h \in {}_p\boldsymbol{J}_K$ such that $\sum_\gamma \mathrm{tr}\,\rho(h^\gamma) \neq 0$. For any such h, 9.5 Corollary asserts that $\boldsymbol{\delta}(\zeta h)$ is a root of unity.

Transfer Lemma. *Let* $h \in {}_p\boldsymbol{J}_K \cap (G_K)^{\mathrm{ell}}_{\mathrm{reg}}$ *and let* $\zeta \in \boldsymbol{\mu}_K$ *be* Γ*-regular. Let*

$$\mathrm{v}_K(h) = \upsilon_K(\det_K h) = \upsilon_F(\det h)/d.$$

Let ϖ_F *be a prime element of* F*, and write* $s = [P{:}K] = ep^r$*. If* $\boldsymbol{\delta}(\zeta h)$ *is a root of unity, then*

$$\boldsymbol{\delta}(\zeta h) = \varkappa(\det h)^{s(d-1)/2} = \varkappa(\varpi_F)^{\mathrm{v}_K(h)n(d-1)/2}.$$

Proof. We use the standard notation for transfer factors, as summarized in [13] §1. The relevant relations are

$$\begin{aligned} \Delta^2(\zeta h) &= \varkappa(e_s\widetilde{\Delta}(\zeta h)), \\ \Delta^1(\zeta h) &= \|\widetilde{\Delta}(\zeta h)^2\|_F^{1/2}\,\|\det \zeta h\|_F^{s(1-d)/2}, \\ \boldsymbol{\delta}(\zeta h) &= \Delta^2(\zeta h)/\Delta^1(\zeta h). \end{aligned} \tag{9.6.3}$$

Here, e_s is a unit in K such that $e_s^\gamma = (-1)^{s(d-1)}e_s$, for a generator γ of Γ. We do not need to give the definition of the function $\widetilde{\Delta}$: we recall only that $\widetilde{\Delta}(\zeta h) \in K^\times$ satisfies $\widetilde{\Delta}(\zeta h)^2 \in F^\times$ and $e_s\widetilde{\Delta}(\zeta h) \in F^\times$.

If $\boldsymbol{\delta}(\zeta h)$ is a root of unity, then $\Delta^1(\zeta h) = 1$ and so

$$\upsilon_F(e_s\widetilde{\Delta}(\zeta h)) = \upsilon_K(\widetilde{\Delta}(\zeta h)) = s(d{-}1)\,\upsilon_F(\det h)/2,$$

whence the lemma follows. $\square$

Proposition. *The character* μ_ξ *of* (9.5.3) *satisfies*

$$\mu_\xi^{\boldsymbol{J}_K}(h) = \varkappa(\det h^{ep^r(d-1)/2}) = \varkappa(\varpi_F)^{\mathrm{v}_K(h)n(d-1)/2}. \tag{9.6.4}$$

Proof. Let $h \in {}_p\boldsymbol{J}_K$. The Linear Independence Lemma of 9.5 gives $u \in J_K^1$ such that $hu \in (G_K)^{\mathrm{ell}}_{\mathrm{reg}}$ and $\sum_\gamma \operatorname{tr}\rho((hu)^\gamma) \neq 0$. The relation (9.6.1) therefore implies $\mu_\xi^{\boldsymbol{J}_K}(hu) = \boldsymbol{\delta}(\zeta hu)$, and this quantity is a root of unity, by 9.5 Corollary. The result now follows from the Transfer Lemma. $\square$

The proposition determines $\mu_\xi^{\boldsymbol{J}_K}$ on ${}_p\boldsymbol{J}_K$. It therefore determines μ_ξ on the group $\det_E(G_E \cap {}_p\boldsymbol{J}_K)$, which contains ϖ_F. We deduce:

Corollary. *Let $\xi \in X_1(E)^{\Delta\text{-reg}}$. The character $\mu_\xi = \xi^{-1}\chi_\xi$ of* (9.5.3) *satisfies*

$$\begin{aligned} \mu_\xi|_{U_E^1} &= 1, \\ \mu_\xi|_{\boldsymbol{\mu}_E} &= \epsilon^1_{\boldsymbol{\mu}_E}, \\ \mu_\xi(\varpi_F) = \varkappa(\varpi_F)^{n(d-1)/2} &= \pm 1, \end{aligned} \tag{9.6.5}$$

for a prime element ϖ_F of F. In particular, if $\xi' \in X_1(E)^{\Delta\text{-reg}}$, then

$$\mu_{\xi'}(z) = \mu_\xi(z), \quad z \in F^\times U_E.$$

In particular, if E_0/F is unramified, then $\mu_\xi = \mu_{\xi'}$.

Proof. Only the third property requires comment. From its definition, the character $\epsilon^1_{\boldsymbol{\mu}_E}$ is trivial on $\boldsymbol{\mu}_F$, so the asserted property is independent of the choice of ϖ_F. We therefore assume ϖ_F is the prime used to define ${}_p\boldsymbol{J}$, that is, ${}_p\boldsymbol{J} = {}_p\boldsymbol{J}(\varpi_F)$. The infinite cyclic group ${}_p\boldsymbol{J}_K/J^1_K$ is therefore generated by an element h_0 of $P^\times$ such that $h_0^{p^r} \equiv \varpi_F \pmod{U^1_P}$. In the notation of the proposition, $\mathrm{v}_K(h_0) = e$. On the other hand, $\det_E(h_0) = \mathrm{N}_{P/E}(h_0) \equiv h_0^{p^r} \pmod{U^1_P}$, since the field extension P/E is totally wildly ramified. This yields

$$\mu_\xi(\varpi_F) = \mu_\xi^{\boldsymbol{J}_K}(h_0) = \varkappa(\varpi_F)^{e^2p^r d(d-1)/2} = \varkappa(\varpi_F)^{n(d-1)/2},$$

as required. □

The corollary completes the proof of the Unramified Induction Theorem in the case where E_0/F is unramified.

Returning to the general case, the character $\epsilon^1_{\boldsymbol{\mu}_K}$ is trivial when $p = 2$, of order $\leqslant 2$ otherwise. Consequently, there is no need to distinguish between $\epsilon^1_{\boldsymbol{\mu}_K}$ and $\epsilon^1_{\boldsymbol{\mu}_K,P}$ as characters of $\boldsymbol{\mu}_K = \boldsymbol{\mu}_E = \boldsymbol{\mu}_P$.

Remark. In (9.6.5), the character $\epsilon^1_F(\boldsymbol{\mu}_E)$ is invariably trivial on $\boldsymbol{\mu}_F$. Consequently, the formula for ϖ_F holds for all prime elements ϖ_F of F, not just the one on which we have chosen to base our constructions. Observe also that $\mu_\xi(\varpi_F) = \pm 1$ and, indeed, takes only the value $+1$ when d is odd.

9.7. We prove the Linear Independence Lemma of 9.5. To summarize the key hypotheses, the representation ρ of G_K is irreducible, cuspidal and totally ramified, the endo-class $\vartheta(\rho) \in \mathcal{E}(K)$ is Θ_K, and E/K is a tame parameter field for Θ_K.

Lemma 1. *The endo-classes $\vartheta(\rho^\gamma)$, $\gamma \in \Delta\backslash\Gamma$, are distinct.*

Proof. Recall that $K \cap E_0 = K_0/F$ is the maximal unramified sub-extension of E_0/F. We think of $\Delta\backslash\Gamma$ as $\mathrm{Gal}(K_0/F)$. We set $\Theta_0 = \mathfrak{i}_{K/K_0}(\Theta_K)$, so that Θ_0 is a K_0/F-lift of Θ. The set of K_0/F-lifts of Θ is therefore the orbit $\{\Theta_0^\gamma : \gamma \in \Delta\backslash\Gamma\}$, the conjugates Θ_0^γ, $\gamma \in \Delta\backslash\Gamma$, being distinct. A tame parameter field for Θ_0 is provided by E_0/K_0. The field extensions K/K_0, E_0/K_0 are linearly disjoint, whence Θ_0^γ has a unique K/K_0-lift, namely $\vartheta(\rho^\gamma)$. □

We temporarily set $s = [P{:}K] = [P_0{:}K_0]$ and $t = [K_0{:}F] = |\Delta\backslash\Gamma|$. We fix a prime element ϖ of P_0 and let $\mathfrak{S}_a = \varpi^a J_K^1 \cap (G_K)_{\mathrm{reg}}^{\mathrm{ell}}$. We recall that $\rho = c\text{-}\mathrm{Ind}_{\boldsymbol{J}_K}^{G_K}\,\kappa_K$. In particular, the central character of ρ is trivial on $\boldsymbol{\mu}_K$ and a pre-determined prime element ϖ_F of F. Consequently:

Lemma 2. *Let $g \in (G_K)_{\mathrm{reg}}^{\mathrm{ell}}$ and suppose $\mathrm{tr}\,\rho(g) \neq 0$. The element g then has a G_K-conjugate lying in $\mu\mathfrak{S}_a$, where $a = \upsilon_K(\det g)$ and μ is some element of $\boldsymbol{\mu}_P = \boldsymbol{\mu}_K$.*

Consider the set of st functions $\mathrm{tr}\,\chi\rho^\gamma$ on $(G_K)_{\mathrm{reg}}^{\mathrm{ell}}$, where γ ranges over $\Delta\backslash\Gamma$ and χ over $X_0(K)_s$. This set is linearly independent. Let $\mathcal{D}$ denote the space it spans.

Each of the representations $\chi\rho^\gamma$ has central character trivial on $\langle\boldsymbol{\mu}_K, \varpi_F\rangle$. Consequently, Lemma 2 implies that their characters form a linearly independent set of functions on the space $\bigcup_{0\leqslant a\leqslant s-1}\mathfrak{S}_a$. Let δ_a denote the dimension of the space $\{f|_{\mathfrak{S}_a} : f \in \mathcal{D}\}$. For fixed γ and a, the functions $\mathrm{tr}\,\chi\rho^\gamma|_{\mathfrak{S}_a}$, $\chi \in X_0(K)_s$, are constant multiples of each other. We conclude that $\delta_a \leqslant t$, for all a. However,

$$st = \dim \mathcal{D} \leqslant \sum_{0\leqslant a\leqslant s-1} \delta_a \leqslant st.$$

It follows that $\delta_a = t$ for all a, whence the functions $\mathrm{tr}\,\rho^\gamma$ are linearly independent on each set $\mathfrak{S}_a$. This proves the Linear Independence Lemma. □

10. Discrepancy at a prime element

We continue our investigation of the character μ_ξ of (9.5.3), attached to a character $\xi \in X_1(E)^{\Delta\text{-reg}}$. We take a particular prime element ϖ of E_0 and show

that the value $\mu_\xi(\varpi)$ is independent of ξ. It will follow that μ_ξ is independent of ξ, as required to finish the proof of 9.2 Theorem. We shall rely on the fact that the proof of that result is already complete when E_0/F is unramified (9.6 Corollary).

10.1. We have chosen in §5 a prime element ϖ_F of F; we take $\varpi \in C_{E_0}(\varpi_F)$ (notation of 5.2). We set $L = F[\varpi]$, and note that E/L is *unramified.* We put $\varUpsilon = \mathrm{Aut}(E|F)$ and $\varUpsilon_L = \mathrm{Gal}(E/L) \subset \varUpsilon$, so that $\varDelta \subset \varUpsilon_L \subset \varUpsilon$. Other notation is as in §9, particularly $d = [K{:}F]$.

We return to the automorphic induction equation as in (9.2.4), but incorporating the conclusion of 9.5 Proposition. It reads

$$\text{(10.1.1)}\quad \mathrm{tr}^{\varkappa}\,\pi_\chi(g) = (-1)^{m-1}\epsilon_F^0(\boldsymbol{\mu}_K)\,\boldsymbol{\delta}_{K/F}(g)\sum_{\gamma\in\varGamma}\mathrm{tr}\,\tau_\xi^\gamma(g),\quad g \in G_K \cap G_{\mathrm{reg}}^{\mathrm{ell}}.$$

(Since we will vary the base field, it is now necessary to specify the field extension to which the transfer factor $\boldsymbol{\delta}$ is attached.) We evaluate each side of (10.1.1) at an element $g = \varpi h$, where h satisfies the following conditions:

(10.1.2) Hypotheses.

(1) $h \in {}_p\boldsymbol{J}_E = {}_p\boldsymbol{J}_E(\varpi_F)$;
(2) *the group* ${}_p\boldsymbol{J}_E$ *is generated by* h *and* J_E^1;
(3) *the element* ϖh *lies in* $G_E \cap G_{\mathrm{reg}}^{\mathrm{ell}}$.

We recall that π_χ contains the extended maximal simple type $\varLambda_\chi = \lambda_\chi^{\boldsymbol{J}} \otimes \kappa \in \mathcal{T}(\theta)$.

Proposition. *Let* $g = \varpi h$*, where* h *satisfies the conditions* (10.1.2). *Let* $x \in G$ *satisfy* $x^{-1}gx \in \boldsymbol{J}$*, and suppose that* $\mathrm{tr}\,\varLambda_\chi(x^{-1}gx) \neq 0$. *The coset* $x\boldsymbol{J}$ *then contains an element* y *for which* $y^{-1}\varpi y \in E^\times$.

Proof. Write $g' = x^{-1}gx$, $g_1' = x^{-1}\varpi x$ and $g_2' = x^{-1}hx$. In the group $G/\langle\varpi_F\rangle$, the element g_1' has finite order relatively prime to p, while g_2' is pro-unipotent. Moreover, the elements g_i' commute with each other. It follows that any closed subgroup of G containing ϖ_F and g' must also contain both g_i'.

Since ϖ is central in $\boldsymbol{J}/J^1$, we may write $g_1' = \varpi h_1$, where $h_1 \in J^0$ has finite order, relatively prime to p, in J^0/J^1. There exists $\varpi' \in P_0^\times \cap {}_p\boldsymbol{J}$ such that

$g_2' = \varpi' h_2$, where $h_2 \in J^0$. This element ϖ' is also central in $\boldsymbol{J}/J^1$. Thus h_2 is unipotent in J^0/J^1, Moreover, the elements h_i commute with each other in J^0/J^1.

We have $g' \equiv h_1 h_2 z \pmod{J^1}$, for some z which is central in $\boldsymbol{J}/J^1$. As $\lambda_\chi^{\boldsymbol{J}}$ is an irreducible representation of $\boldsymbol{J}$, trivial on J^1, we deduce that $\operatorname{tr}\lambda_\chi^{\boldsymbol{J}}(g') \neq 0$ if and only if $\operatorname{tr}\lambda_\chi^{\boldsymbol{J}}(h_1h_2) \neq 0$.

As in the proof of 6.1 Proposition 12 of [13], the condition $\operatorname{tr}\lambda_\chi^{\boldsymbol{J}}(h_1h_2) \neq 0$ implies that h_1 is conjugate in J^0/J^1 to an element of $\boldsymbol{\mu}_K$. We may therefore adjust x within the coset $x\boldsymbol{J}$ in order to assume that $h_1 = \alpha_1 j$, where $\alpha_1 \in \boldsymbol{\mu}_K$ and $j \in J^1$. The element $g_1' = \varpi h_1$ acts on J^1 (by conjugation) as an automorphism of finite order, relatively prime to p. Applying the Conjugacy Lemma of 2.6, we may further adjust x by an element of J^1 to assume that j commutes with $h_1\varpi$. However, $\alpha_1\varpi j = g_1'$ has finite p-prime order in $G/\langle\varpi_F\rangle$. It follows that $j=1$, whence $g_1' \in E^\times$. $\square$

The extension E/L is unramified. Consequently, any F-embedding $L \to E$ extends to an F-automorphism of E. The set of $x \in G$ for which $x^{-1}\varpi x \in E$ is therefore $G_L N_E$, where N_E denotes the G-normalizer of $E^\times$. Thus N_E/G_E is canonically isomorphic to $\varUpsilon = \operatorname{Aut}(E|F)$.

We choose a section $\alpha \mapsto g_\alpha$ of the canonical map $N_E/G_E \to \varUpsilon$, such that $g_\alpha \in J^0$ when $\alpha \in \varDelta$. The coset space $G_L N_E \boldsymbol{J}/\boldsymbol{J}$ can then be decomposed as a disjoint union

$$G_L N_E \boldsymbol{J}/\boldsymbol{J} = \bigcup_\alpha G_L g_\alpha \boldsymbol{J}/\boldsymbol{J} = \bigcup_\alpha g_\alpha G_{L^\alpha}\boldsymbol{J}/\boldsymbol{J} = \bigcup_\alpha g_\alpha G_{L^\alpha}/\boldsymbol{J}_{L^\alpha},$$

where α ranges over $\varUpsilon_L\backslash\varUpsilon/\varDelta$. Here, we view $\varDelta$ as $\operatorname{Gal}(E/E_0)$ and note that $\boldsymbol{J}_{L^\alpha}$ is not necessarily the same as $(\boldsymbol{J}_L)^\alpha$. We accordingly decompose the $\varkappa$-trace

$$\operatorname{tr}^\varkappa \pi_\chi(g) = \sum_{\alpha\in\varUpsilon_L\backslash\varUpsilon/\varDelta} \mathcal{L}_\chi(g;\alpha), \tag{10.1.3}$$

where

$$\mathcal{L}_\chi(g;\alpha) = \sum_{y\in G_L^\alpha/\boldsymbol{J}_{L^\alpha}} \varkappa(\det y^{-1})\operatorname{tr}\varLambda_\chi(y^{-1}g^\alpha y). \tag{10.1.4}$$

Remark. For $\alpha \in \varUpsilon$, the field $E_0^\alpha \subset E$ is isomorphic to a tame parameter field for θ. However, the decomposition of (10.1.3) and 2.6 Proposition show that it will not be a tame parameter field unless it equals E_0.

10.2. We return to (10.1.1) and write

$$\mathcal{R}(g) = (-1)^{m-1}\epsilon_F^0(\boldsymbol{\mu}_K)\boldsymbol{\delta}_{K/F}(g)\sum_{\gamma\in\Gamma}\operatorname{tr}\tau_\xi^\gamma(g), \quad g\in G_K\cap G_{\mathrm{reg}}^{\mathrm{ell}}.$$

We analyze this expression at the element $g=\varpi h$, where $h\in G_E$ satisfies (10.1.2).

Lemma 1. *Let $\gamma\in\Gamma$ and suppose that $\operatorname{tr}\tau^\gamma(\varpi h)\neq 0$. The automorphism γ of K then extends to an F-automorphism of E.*

Proof. By hypothesis, $\varpi h\in G_E\cap G_{\mathrm{reg}}^{\mathrm{ell}}$. The algebra $F[\varpi h]$ is therefore a field, of degree n over F. It commutes with, and therefore contains, K. Condition (10.1.2)(2) implies $F[\varpi h]/K$ to be totally ramified, of degree n/d. The argument is now identical to that of 8.4 Lemma. $\square$

Lemma 1 implies

$$\mathcal{R}(\varpi h) = (-1)^{m-1}\,\epsilon_F^0(\boldsymbol{\mu}_K)\,\boldsymbol{\delta}_{K/F}(\varpi h)\sum_{\gamma\in\Upsilon/\Upsilon_K}\operatorname{tr}\tau_\xi^\gamma(\varpi h),$$

where Υ_K is the subgroup $\operatorname{Aut}(E|K)$ of Υ. We take $\kappa\in\mathcal{H}(\theta)$, as in 9.2. We form $\kappa_K=\ell_{K/F}(\kappa)\in\mathcal{H}(\theta_K)^\Delta$, as in 5.6 Proposition, and then $\kappa_E=(\kappa_K)_E\in\mathcal{H}(\theta_E)=\mathcal{T}(\theta_E)$, as in 5.3 Proposition. We set

$$\rho = c\text{-}\operatorname{Ind}_{\boldsymbol{J}_E}^{G_E}\kappa_E. \tag{10.2.1}$$

Thus $\tau_\xi\cong ind_{E/K}\,\xi\rho$ and, equivalently, $(\tau_\xi)_E=\xi\rho$. Applying 8.1 Proposition, relative to the base field K, we get

$$\operatorname{tr}\tau_\xi(\varpi h)=\epsilon_K(\varpi)\sum_{\beta\in\Upsilon_K}\operatorname{tr}(\xi\rho)^\beta(\varpi h),$$

where $\epsilon_K(\varpi)$ is the symplectic sign $t_{\langle\varpi\rangle}(J_K^1/H_K^1)$. In all,

$$\mathcal{R}(\varpi h) = (-1)^{m-1}\,\epsilon_F^0(\boldsymbol{\mu}_K)\,\boldsymbol{\delta}_{K/F}(\varpi h)\sum_{\gamma\in\Upsilon}\epsilon_K(\varpi^\gamma)\operatorname{tr}(\xi\rho)^\gamma(\varpi h).$$

The element ϖ^γ/ϖ lies in $\boldsymbol{\mu}_K$, and so acts trivially on J_K^1/H_K^1. Therefore $\epsilon_K(\varpi^\gamma)=\epsilon_K(\varpi)$ and so

$$\mathcal{R}(\varpi h) = (-1)^{m-1}\,\epsilon_F^0(\boldsymbol{\mu}_K)\,\epsilon_K(\varpi)\,\boldsymbol{\delta}_{K/F}(\varpi h)\sum_{\gamma\in\Upsilon}\operatorname{tr}(\xi\rho)^\gamma(\varpi h). \tag{10.2.2}$$

We write the inner sum here in the form $\sum_{\psi\in\Upsilon} \operatorname{tr}\xi\rho(\varpi^\psi h^\psi)$, and consider an individual term $\operatorname{tr}\xi\rho(\varpi^\psi h^\psi)$. By construction, the central character of ρ is trivial on $C_E(\varpi_F)$, so

$$\operatorname{tr}\xi\rho(\varpi^\psi h^\psi) = \xi(\det_E(\varpi^\psi h^\psi))\operatorname{tr}\rho(h^\psi).$$

The element $h \in {}_p\boldsymbol{J}_E$ satisfies $h^{p^r} \equiv \varpi_F^a \pmod{J_E^1}$, for an integer a relatively prime to $p^r e$. It follows that $\det_E h \equiv \varpi_F^a \pmod{U_E^1}$, whence $\xi(\det_E h^\psi) = \xi(\varpi_F)^a$. For the other factor, there is a root of unity $\zeta_\psi \in \boldsymbol{\mu}_E$ such that $\varpi^\psi = \zeta_\psi\varpi$, giving $\xi(\det_E \varpi^\psi) = \xi(\zeta_\psi)^{p^r}\,\xi(\varpi)^{p^r}$. We therefore have

$$\begin{aligned}(10.2.3)\quad \mathcal{R}(\varpi h) &\\ &= (-1)^{m-1}\epsilon_F^0(\boldsymbol{\mu}_E)\,\epsilon_K(\varpi)\,\boldsymbol{\delta}_{K/F}(\varpi h)\,\xi(\varpi_F^a\varpi^{p^r}) \\ &\qquad\qquad \sum_{\psi\in\Upsilon}\xi(\zeta_\psi)^{p^r}\operatorname{tr}\rho(h^\psi).\end{aligned}$$

Before passing on, we record some properties of the root of unity ζ_ψ.

Lemma 2.

(1) *If $\gamma \in \Upsilon_L$, then $\zeta_\gamma = 1$. In particular, $\zeta_\delta = 1$ if $\delta \in \Delta$.*
(2) *If $\psi = \gamma\alpha\delta$, $\gamma\in\Upsilon_L$, $\delta\in\Delta$, then $\zeta_\psi = \zeta_\alpha^\delta$.*

Proof. If $\beta \in \Upsilon_L$, then $\varpi^\beta = \varpi$ by definition, and this proves (1). The map $\psi \mapsto \zeta_\psi$ satisfies $\zeta_{\psi\phi} = \zeta_\psi^\phi\zeta_\phi$, $\psi,\phi\in\Upsilon$, whence (2) follows. □

The representation ρ is fixed by Δ. Indeed, Δ is the Υ-isotropy group of ρ. Using Lemma 2, we may now re-write (10.2.3) in the form

$$\begin{aligned}(10.2.4)\quad \mathcal{R}(\varpi h) &\\ &= (-1)^{m-1}\epsilon_F^0(\boldsymbol{\mu}_E)\,\epsilon_K(\varpi)\,\boldsymbol{\delta}_{K/F}(\varpi h)\,\xi(\varpi_F^a\varpi^{p^r}) \\ &\qquad\qquad \sum_{\psi\in\Upsilon/\Delta}\operatorname{tr}\rho(h^\psi)\sum_{\delta\in\Delta}\xi(\zeta_\psi^{p^r})^\delta.\end{aligned}$$

10.3. We return to the expression (10.1.4),

$$\mathcal{L}_\chi(\varpi h;\alpha) = \sum_{y\in G_L^\alpha/\boldsymbol{J}_{L^\alpha}} \varkappa(\det y^{-1})\operatorname{tr}\Lambda_\chi(y^{-1}\varpi^\alpha h^\alpha y),$$

for $\alpha \in \Upsilon_L\backslash\Upsilon/\Delta$. Recalling that $\Lambda_\chi = \lambda_\chi^{\boldsymbol{J}} \otimes \kappa$, we first have

$$\operatorname{tr}\lambda_\chi^{\boldsymbol{J}}(\varpi^\alpha y^{-1}h^\alpha y) = \chi(\varpi)^{p^r}\operatorname{tr}\lambda_\chi^{\boldsymbol{J}}(\zeta_\alpha y^{-1}h^\alpha y). \tag{10.3.1}$$

To evaluate the expression (10.3.1), we first use the method of [13] 6.10. We note that E_0L^α/L^α is a tame parameter field for θ_{L^α} and $\mathrm{Gal}(E/E_0L^\alpha) = \Delta\cap\Upsilon_L^\alpha$. We put $m_\alpha = [E{:}E_0L^\alpha]$. Lemma 6.10 of [13] then yields

$$\operatorname{tr}\lambda_\chi^{\boldsymbol{J}}(\zeta_\alpha y^{-1}h^\alpha y) = (-1)^{m-m_\alpha}\sum_{\beta\in\Delta\cap\Upsilon_L^\alpha\backslash\Delta}\operatorname{tr}\lambda_{\chi^\beta}^{\boldsymbol{J}_{L^\alpha}}(\zeta_\alpha y^{-1}h^\alpha y). \tag{10.3.2}$$

Second, we use the Glauberman correspondence relative to the action of the element ϖ^α of $C_E(\varpi_F)$. We obtain a representation $\kappa_{L^\alpha} \in \mathcal{H}(\theta_{L^\alpha})$ and a constant $\epsilon_F(\varpi^\alpha) = t_{\langle\varpi^\alpha\rangle}(J^1/H^1) = \pm 1$ such that

$$\operatorname{tr}\kappa(\varpi^\alpha y^{-1}h^\alpha y) = \epsilon_F(\varpi^\alpha)\operatorname{tr}\kappa_{L^\alpha}(y^{-1}h^\alpha y). \tag{10.3.3}$$

For $\beta\in\Delta$, we set

$$\sigma_{\alpha,\beta} = c\text{-}\mathrm{Ind}_{\boldsymbol{J}_{L^\alpha}}^{G_{L^\alpha}}\lambda_{\chi^\beta}^{\boldsymbol{J}_{L^\alpha}}\otimes\kappa_{L^\alpha} \in \mathcal{A}_{m_\alpha}^0(L^\alpha;\Theta_{L^\alpha}),$$

where $\Theta_{L^\alpha} = cl(\theta_{L^\alpha})$. Setting $\varkappa_{L^\alpha} = \varkappa_L^\alpha = \varkappa\circ\mathrm{N}_{L^\alpha/F}$, the relations (10.3.2), (10.3.3) yield

$$\mathcal{L}_\chi(\varpi h;\alpha) = (-1)^{m-m_\alpha}\,\epsilon_F(\varpi^\alpha)\sum_{\beta\in\Delta\cap\Upsilon_L^\alpha\backslash\Delta}\operatorname{tr}^{\varkappa_L^\alpha}\sigma_{\alpha,\beta}(\varpi^\alpha h^\alpha).$$

We may now form the representation $\ell_{E/L^\alpha}(\kappa_{L^\alpha}) \in \mathcal{H}(\theta_E)$. Transitivity of the Glauberman correspondence (5.1.1) implies that $\ell_{E/L^\alpha}(\kappa_{L^\alpha})$ is equivalent to the representation κ_E defined in 10.2. As there, we set $\rho = c\text{-}\mathrm{Ind}_{\boldsymbol{J}_E}^{G_E}\kappa_E$. It follows that $\sigma_{\alpha,\beta}$ is automorphically induced from a representation $\chi^\beta\eta_{\alpha,\beta}\cdot\rho$ of G_E, where $\eta_{\alpha,\beta}\in X_1(E)$ is given by 9.6 Corollary. The explicit formula (9.6.5) shows $\eta_{\alpha,\beta}$ depends only on the subfield L^α of P, so we write $\eta_{\alpha,\beta} = \eta_\alpha$. Therefore

$$\begin{aligned}&\operatorname{tr}^{\varkappa_L^\alpha}\sigma_{\alpha,\beta}(\varpi^\alpha h^\alpha)\\ &\qquad = (-1)^{m_\alpha-1}\,\epsilon_{L^\alpha}^0(\boldsymbol{\mu}_E)\,\boldsymbol{\delta}_{E/L^\alpha}(\varpi^\alpha h^\alpha)\sum_{\delta\in\Upsilon_L^\alpha}\operatorname{tr}(\chi^\beta\eta_\alpha\cdot\rho)^\delta(\varpi^\alpha h^\alpha).\end{aligned}$$

The definition gives $\boldsymbol{\delta}_{E/L^\alpha}(\varpi^\alpha h^\alpha) = \boldsymbol{\delta}_{E/L}(\varpi h)$. The representations η_α, ρ are fixed by Δ and, in particular, by the element β. So, adding up, we get

$$\begin{aligned}&\mathcal{L}_\chi(\varpi h;\alpha)\\ &\quad = (-1)^{m-1}\,\epsilon_F(\varpi^\alpha)\,\epsilon_{L^\alpha}^0(\boldsymbol{\mu}_E)\,\boldsymbol{\delta}_{E/L}(\varpi h)\sum_{\psi\in\Upsilon_L\alpha\Delta}\operatorname{tr}(\chi\eta_\alpha\cdot\rho)((\varpi h)^\psi).\end{aligned} \tag{10.3.4}$$

10.4. In (10.3.4), we consider the term

$$\operatorname{tr}(\chi\eta_\alpha\cdot\rho)((\varpi h)^\psi) = \chi(\det_E(\varpi h)^\psi)\,\eta_\alpha(\det_E(\varpi h)^\psi)\operatorname{tr}\rho(h^\psi),$$

for $\psi \in \varUpsilon_L\alpha\varDelta$. The element h lies in ${}_p\boldsymbol{J}(\varpi_F)$, and satisfies $h^{p^r} \equiv \varpi_F^a \pmod{J^1}$. This implies $\det_E h \equiv \varpi_F^a \pmod{U_E^1}$, and therefore

$$\chi(\det_E h^\psi) = \chi(\varpi_F^a). \tag{10.4.1}$$

Also,

$$\chi(\det_E \varpi^\psi) = \chi(\varpi^\psi)^{p^r} = \chi(\zeta_\psi^{p^r})\,\chi(\varpi)^{p^r}, \tag{10.4.2}$$

where $\zeta_\psi = \varpi^\psi/\varpi \in \boldsymbol{\mu}_K$, as in 10.2.

We next consider the contributions $\eta_\alpha(\det_E(\varpi h)^\psi)$. We write $\psi = \gamma\alpha\delta$, with $\gamma \in \varUpsilon_L$ and $\delta \in \varDelta$. It will be useful to have the notation

$$d_L = [E{:}L] = [E{:}L^\alpha],$$

and to recall that $e = e(E|F)$. By (9.6.5),

$$\eta_\alpha(\varpi^\alpha) = \varkappa_L^\alpha(\varpi^\alpha)^{[P:L](d_L-1)/2} = \varkappa(\varpi_F)^{n(d_L-1)/2e}.$$

Therefore

$$\eta_\alpha(\det_E(\varpi^\alpha)) = \varkappa(\varpi_F)^{p^r n(d_L-1)/2e}.$$

Since $(\varpi^\alpha)^\delta = \zeta_{\alpha\delta}\zeta_\alpha^{-1}\varpi^\alpha = \zeta_\alpha^\delta\zeta_\alpha^{-1}\varpi^\alpha$, we get

$$\begin{aligned}\eta_\alpha(\det_E(\varpi^\psi)) &= \eta_\alpha(\det_E(\varpi^{\alpha\delta}))\\ &= \epsilon^1_{L^\alpha}(\boldsymbol{\mu}_E, \zeta_\alpha^\delta\zeta_\alpha^{-1})\,\varkappa(\varpi_F)^{p^r n(d_L-1)/2e}.\end{aligned}$$

All ϵ^1-characters of $\boldsymbol{\mu}_K$ have order $\leqslant 2$, and so are fixed by all automorphisms of $\boldsymbol{\mu}_K$. Therefore

$$\eta_\alpha(\det_E(\varpi^\psi)) = \varkappa(\varpi_F)^{p^r n(d_L-1)/2e}. \tag{10.4.3}$$

The next term to consider is $\eta_\alpha(\det_E h^\psi) = \eta_\alpha(\varpi_F^a)$. We have $\varpi^e = \nu\varpi_F$, for some $\nu \in \boldsymbol{\mu}_L$, whence $(\varpi^\alpha)^e = \nu^\alpha\varpi_F$ and $\nu^\alpha \in \boldsymbol{\mu}_{L^\alpha}$. In particular, $\epsilon^1_{L^\alpha}(\boldsymbol{\mu}_E;\nu^\alpha) = 1$. This gives us

$$\eta_\alpha(\varpi_F) = \eta_\alpha(\varpi^\alpha)^e = \kappa(\varpi_F)^{n(d_L-1)/2},$$

and so

$$\eta_\alpha(\det_E h^\psi) = \eta_\alpha(\varpi_F^a) = \varkappa(\varpi_F)^{an(d_L-1)/2}. \tag{10.4.4}$$

Incorporating (10.4.1–4) into (10.3.4), it becomes

$$\begin{aligned}(10.4.5)\quad & \mathcal{L}_\chi(\varpi h;\alpha) \\ &= (-1)^{m-1}\,\boldsymbol{\delta}_{E/L}(\varpi h)\,\chi(\varpi_F^a\varpi^{p^r})\,\varkappa(\varpi_F)^{(p^rn/e+an)(d_L-1)/2} \\ &\qquad \sum_{\psi\in\Upsilon_L\alpha\Delta} \epsilon_F(\varpi^\alpha)\,\epsilon^0_{L^\alpha}(\boldsymbol{\mu}_E)\,\chi(\zeta_\psi^{p^r})\,\mathrm{tr}\,\rho(h^\psi).\end{aligned}$$

The factor $\chi(\zeta_\psi)$ is $\xi(\zeta_\psi)\epsilon^1_F(\boldsymbol{\mu}_E;\zeta_\psi)$ (9.6 Corollary). If $\psi = \gamma\alpha\delta$, we have $\zeta_\psi = \zeta_\alpha^\delta$, and so

$$\chi(\zeta_\psi)^{p^r} = \xi(\zeta_\psi)^{p^r}\,\epsilon^1_F(\boldsymbol{\mu}_E;\zeta_\alpha).$$

We introduce a new function

$$\mathsf{k}(h) = \varkappa(\varpi_F)^{(p^rn/e+an)(d_L-1)/2} = \pm 1,$$

where $a = a(h)$ is the integer defined by $h^{p^r} \equiv \varpi_F^a \pmod{J^1}$.

Remark. The quantity $\mathsf{k}(h)$ is the product of the constant $\varkappa(\varpi_F)^{p^rn(d_L-1)/2} = \pm 1$ and the character ${}_p\boldsymbol{J}_E \to \{\pm 1\}$ given by $h \mapsto \varkappa(\varpi_F)^{an(d_L-1)/2}$, where $a = a(h)$ is defined by $h^{p^r} \equiv \det_E h \equiv \varpi_F^a \pmod{J^1}$. This character is non-trivial exactly when d_L is even and $ep^r = n/d$ is odd.

We return to the automorphic induction relation (10.1.1) in the form

$$\mathcal{R}(g) = \sum_{\alpha\in\Upsilon_L\backslash\Upsilon/\Delta} \mathcal{L}_\chi(g;\alpha).$$

Following (10.2.3) and (10.4.5), it becomes

$$\begin{aligned}(10.4.6)\quad & \epsilon^0_F(\boldsymbol{\mu}_E)\epsilon_K(\varpi)\,\xi(\varpi_F^a\varpi^{p^r})\,\boldsymbol{\delta}_{K/F}(\varpi h)\sum_{\psi\in\Upsilon}\xi(\zeta_\psi)^{p^r}\,\mathrm{tr}\,\rho(h^\psi) \\ &= \mathsf{k}(h)\,\chi(\varpi_F^a\varpi^{p^r})\,\boldsymbol{\delta}_{E/L}(\varpi h)\sum_{\alpha\in\Upsilon_L\backslash\Upsilon/\Delta}\epsilon_F(\varpi^\alpha)\,\epsilon^0_{L^\alpha}(\boldsymbol{\mu}_E)\,\epsilon^1_F(\boldsymbol{\mu}_E;\zeta_\alpha) \\ &\qquad \sum_{\psi\in\Upsilon_L\alpha\Delta}\xi(\zeta_\psi)^{p^r}\,\mathrm{tr}\,\rho(h^\psi).\end{aligned}$$

By (9.6.5), we have

$$\chi(\varpi_F^a) = \xi(\varpi_F^a)\,\varkappa(\varpi_F)^{an(d-1)/2}. \tag{10.4.7}$$

To simplify further, we need to control the coefficients in the sum. This is achieved via:

Symplectic Signs Lemma. *The function* $\varUpsilon_L\backslash\varUpsilon/\varDelta \to \{\pm 1\}$, *given by*

$$\varUpsilon_L\alpha\varDelta \longmapsto \epsilon_F(\varpi^\alpha)\,\epsilon^0_{L^\alpha}(\boldsymbol{\mu}_E)\,\epsilon^1_F(\boldsymbol{\mu}_E;\zeta_\alpha),$$

is constant.

We prove this lemma in §11 below. In particular, it implies

$$\epsilon_F(\varpi^\alpha)\,\epsilon^0_{L^\alpha}(\boldsymbol{\mu}_E)\,\epsilon^1_F(\boldsymbol{\mu}_E;\zeta_\alpha) = \epsilon_F(\varpi)\,\epsilon^0_L(\boldsymbol{\mu}_E), \quad \alpha\in\varUpsilon. \tag{10.4.8}$$

We accordingly set

$$\begin{aligned} \mathsf{k}'(h) &= \epsilon_K(\varpi)\,\epsilon_F(\varpi)\,\epsilon^0_L(\boldsymbol{\mu}_E)\,\epsilon^0_F(\boldsymbol{\mu}_E)\,\varkappa(\varpi_F)^{an(d-1)/2}\,\mathsf{k}(h) \\ &= \epsilon_K(\varpi)\,\epsilon_F(\varpi)\,\epsilon^0_L(\boldsymbol{\mu}_E)\,\epsilon^0_F(\boldsymbol{\mu}_E)\,\varkappa(\varpi_F)^{p^rn(d_L-1)/2e+an(d-d_L)/2} \\ &= \pm 1. \end{aligned} \tag{10.4.9}$$

Remark. As for the function k, the quantity $\mathsf{k}'(h)$ is the product of a constant, with value ±1, and the character ${}_p\boldsymbol{J}_E \to \{\pm 1\}$ given by $h\mapsto \varkappa(\varpi_F)^{an(d-d_L)/2}$. This character is non-trivial exactly when $d\not\equiv d_L\equiv ep^r\equiv 1 \pmod 2$.

The fundamental relation (10.4.6) so becomes

$$\begin{aligned} \xi(\varpi^{p^r})\,\boldsymbol{\delta}_{K/F}(\varpi h) &\sum_{\psi\in\varUpsilon/\varDelta} \operatorname{tr}\rho(h^\psi) \sum_{\delta\in\varDelta} \xi(\zeta^\delta_\psi)^{p^r} \\ &= \mathsf{k}'(h)\,\chi(\varpi^{p^r})\,\boldsymbol{\delta}_{E/L}(\varpi h) \sum_{\psi\in\varUpsilon/\varDelta} \operatorname{tr}\rho(h^\psi) \sum_{\delta\in\varDelta} \xi(\zeta^\delta_\psi)^{p^r} \end{aligned} \tag{10.4.10}$$

(*cf.* (10.2.4)).

10.5. To get any further with (10.4.10), we have to cancel the double sum occurring in each side.

Linear Independence Lemma. *Let h satisfy* (10.1.2), *and let S_h be the set of $u\in J^1_E$ for which hu satisfes the same conditions. The set of functions $u\mapsto \operatorname{tr}\rho^\psi(hu)$, $\psi\in\varDelta\backslash\varUpsilon$, is then linearly independent on S_h.*

Proof. Let S'_h be the set of $u\in J^1_E$ for which $\varpi hu\in (G_E)^{\mathrm{ell}}_{\mathrm{reg}}$. The set of functions $u\mapsto \operatorname{tr}\rho^\psi(hu)$, $\psi\in\varDelta\backslash\varUpsilon$, is then linearly independent on S'_u: the argument is the same as the proof of the Linear Independence Lemma of 9.5. However, these functions are all locally constant on S'_h, and S_h is dense in S'_h. $\square$

Non-vanishing Lemma. *Let h satisfy* (10.1.2), *and define S_h as in the Linear Independence Lemma. Let $\xi \in X_1(E)^{\Delta\text{-reg}}$. There exists $u \in S_h$ such that*

$$\sum_{\psi \in \Upsilon/\Delta} \operatorname{tr} \rho((hu)^\psi) \sum_{\delta \in \Delta} \xi(\zeta_\psi^\delta)^{p^r} \neq 0.$$

Proof. In light of the Linear Independence Lemma, it is enough to show that one of the coefficients $\sum_\delta \xi(\zeta_\psi^\delta)^{p^r}$, $\psi \in \Upsilon/\Delta$, is non-zero. However, if we take $\psi = 1$, this coefficient reduces to $|\Delta| = m$. □

The character $\xi^{-1}\chi$ has finite order (9.5 Corollary), so we conclude:

Proposition. *Let $\xi \in X_1(E)^{\Delta\text{-reg}}$, Let $\chi = \chi_\xi$ and let h satisfy* (10.1.2). *There exists $u \in S_h$ such that*

(10.5.1) $$\xi^{-1}\chi(\varpi^{p^r}) = \Bbbk'(hu)\, \boldsymbol{\delta}_{K/F}(\varpi hu)/\boldsymbol{\delta}_{E/L}(\varpi hu).$$

In particular, $\Bbbk'(hu) = \Bbbk'(h) = \pm 1$, and $\boldsymbol{\delta}_{K/F}(\varpi hu)/\boldsymbol{\delta}_{E/L}(\varpi hu)$ is a root of unity.

10.6. To reduce the notational load, we assume h was chosen so that 10.5 Proposition holds with $u = 1$: this is certainly achievable but the choice of h may still depend on ξ.

Transfer Lemma. *If the quantity $\boldsymbol{\delta}_{K/F}(\varpi h)/\boldsymbol{\delta}_{E/L}(\varpi h)$ is a root of unity, then*

$$\boldsymbol{\delta}_{K/F}(\varpi h)/\boldsymbol{\delta}_{E/L}(\varpi h) = \mathrm{d}\, \varkappa(\varpi_F)^{an(d-d_L)/2},$$

where a is the integer defined by $h^{p^r} \equiv \varpi_F^a \pmod{J^1}$, and d is a sign depending only on the ramification indices and inertial degrees in the tower of fields $F \subset L \subset P$.

Proof. We set $t = \varpi h$ and return to the defining formulas

$$\Delta^1_{K/F}(t) = \|\widetilde{\Delta}_{K/F}(t)^2\|_F^{1/2}\, \|\det t\|_F^{ep^r(1-d)/2},$$

$$\Delta^1_{E/L}(t) = \|\widetilde{\Delta}_{E/L}(t)^2\|_L^{1/2}\, \|\det_L t\|_L^{p^r(1-d_L)/2},$$

$$\Delta^2_{K/F}(t) = \varkappa(e_F \widetilde{\Delta}_{K/F}(t)), \qquad \Delta^2_{E/L}(t) = \varkappa_L(e_L \widetilde{\Delta}_{E/L}(t)),$$

$$\boldsymbol{\delta}_{K/F}(t) = \Delta^2_{K/F}(t)/\Delta^1_{K/F}(t), \quad \boldsymbol{\delta}_{E/L}(t) = \Delta^2_{E/L}(t)/\Delta^1_{E/L}(t).$$

The elements e_F, e_L are units of K, E respectively, such that $e_F\widetilde{\Delta}_{K/F}(t) \in F^\times$ and $\widetilde{\Delta}_{E/L}(t) \in L^\times$. Both are independent of t.

As $\boldsymbol{\delta}_{K/F}(t)/\boldsymbol{\delta}_{E/L}(t)$ is a root of unity, it follows that $\Delta^1_{K/F}(t) = \Delta^1_{E/L}(t)$, or that

$$\frac{\|\widetilde{\Delta}_{K/F}(t)^2\|_F}{\|\det t\|_F^{ep^r(d-1)}} = \frac{\|\widetilde{\Delta}_{E/L}(t)^2\|_L}{\|{\det}_L\, t\|_L^{p^r(d_L-1)}} = \frac{\|\mathrm{N}_{L/F}\,\widetilde{\Delta}_{E/L}(t)^2\|_F}{\|\det t\|_F^{p^r(d_L-1)}}. \tag{10.6.1}$$

Let $\mathrm{v}_{K/F}$ denote $\upsilon_K(\widetilde{\Delta}_{K/F}(t))$, so that $\Delta^2_{K/F}(t) = \varkappa(\varpi_F)^{\mathrm{v}_{K/F}(t)}$. Let $\mathrm{v}_{E/L} = \upsilon_E(\widetilde{\Delta}_{E/L}(t))$. Thus, if ϖ_L is a prime element of L, then

$$\Delta^2_{E/L}(t) = \varkappa_L(\varpi_L)^{\mathrm{v}_{E/L}} = \varkappa(\varpi_F)^{d\mathrm{v}_{E/L}/d_L}.$$

Moreover,

$$\boldsymbol{\delta}_{K/F}(t)/\boldsymbol{\delta}_{E/L}(t) = \Delta^2_{K/F}(t)/\Delta^2_{E/L}(t) = \varkappa(\varpi_F)^{\mathrm{v}_{K/F}-d\mathrm{v}_{E/L}/d_L}. \tag{10.6.2}$$

If $\mathrm{v}_0 = \upsilon_F(\det t)$, the relation (10.6.1) implies

$$\mathrm{v}_{K/F} - d\mathrm{v}_{E/L}/d_L = \mathrm{v}_0\big(ep^r(d{-}1)/2 - p^r(d_L{-}1)/2\big).$$

However, $\mathrm{v}_0 = ade + p^r d$, giving

$$\begin{aligned}\mathrm{v}_{K/F} - d\mathrm{v}_{E/L}/d_L &= \big(ade{+}p^r d/d_L\big)\big(ep^r(d{-}1)/2 - p^r(d_L{-}1)/2\big)\\ &= p^r\bigg(ade^2(d{-}1)/2 - ade(d_L{-}1)/2\\ &\qquad\qquad + ep^r d(d{-}1)/2 - p^r d(d_L{-}1)/2\bigg).\end{aligned}$$

The third and fourth terms contribute a constant integer

$$w = p^{2r}d\big(e(d{-}1) - (d_L{-}1)\big)/2,$$

depending only on the fields $F \subset L \subset P$. Since $\varkappa(\varpi_F)$ is a primitive d-th root of unity and $ep^r d = n$, the relation (10.6.2) reduces to

$$\boldsymbol{\delta}_{K/F}(t)/\boldsymbol{\delta}_{E/L}(t) = \mathrm{d}\,\varkappa(\varpi_F)^{an(d-d_L)/2},$$

where $\mathrm{d} = \varkappa(\varpi_F)^w = \pm 1$. $\square$

We combine the conclusion of the Transfer Lemma with (10.4.9) and (10.5.1) to get

$$\xi^{-1}\chi(\varpi)^{p^r} = \mathrm{d}'\,\epsilon_K(\varpi)\,\epsilon_F(\varpi)\,\epsilon^0_L(\boldsymbol{\mu}_E)\,\epsilon^0_F(\boldsymbol{\mu}_E), \tag{10.6.3}$$

where d' is a sign depending only on the ramification indices and inertial degrees in the tower of fields $F \subset L \subset P$. By 9.6 Corollary, $\xi^{-1}\chi(\varpi)$ is an e-th root of ± 1. Since e is relatively prime to p, there is a unique character μ_ξ of $E^\times$ satisfying (9.6.5) and

$$\mu_\xi(\varpi)^{p^r} = \mathrm{d}'\,\epsilon_K(\varpi)\,\epsilon_F(\varpi)\,\epsilon^0_L(\boldsymbol{\mu}_E)\,\epsilon^0_F(\boldsymbol{\mu}_E).$$

This is visibly independent of $\xi \in X_1(E)^{\Delta-\mathrm{reg}}$ and is fixed by Δ.

We have completed the proof of 9.2 Theorem and, with it, the proof of the Unramified Induction Theorem of 9.1. □

10.7. For convenience, we summarize our conclusions concerning the character $\mu = \mu_\alpha^{K/F}$ of 9.2 Theorem.

Corollary.

(1) *The character* $\mu = \mu_\alpha^{K/F} \in X_1(E)^\Delta$ *satisfies:*

$$\begin{aligned} \mu|_{U^1_E} &= 1, \\ \mu|_{\boldsymbol{\mu}_E} &= \epsilon^1_{\boldsymbol{\mu}_E} = t^1_{\boldsymbol{\mu}_E}(J^1_\theta/H^1_\theta), \\ \mu(\varpi_F) &= \varkappa(\varpi_F)^{n(d-1)/2} = \pm 1, \end{aligned}$$

where ϖ_F *is a prime element of* F.

(2) *If* ϖ *is a prime element of* E_0 *lying in* $C_{E_0}(\varpi_F)$, *then*

$$\mu(\varpi)^{p^r} = \mathrm{d}'\,\epsilon_K(\varpi)\,\epsilon_F(\varpi)\,\epsilon^0_L(\boldsymbol{\mu}_E)\,\epsilon^0_F(\boldsymbol{\mu}_E),$$

where d' *is a sign depending only on the ramification indices and inertial degrees in the tower of fields* $F \subset L = F[\varpi] \subset P$.

(3) *These conditions determine* μ *uniquely, and* μ *takes its values in* $\{\pm 1\}$.

Proof. Parts (1) and (2) have already been done. Setting $e = e(E|F)$, we have $\varpi^e = \zeta\varpi_F$, for some $\zeta \in \boldsymbol{\mu}_{E_0}$. We choose integers a and b so that $ae + bp^r = 1$. We get

$$\mu(\varpi) = \big(\epsilon^1_{\boldsymbol{\mu}_E}(\zeta)\mu(\varpi_F)\big)^a \cdot \mu(\varpi)^{bp^r} = \pm 1,$$

whence all outstanding assertions follow. □

11. Symplectic signs

We prove the Symplectic Signs Lemma of 10.4. This concerns the sign invariants attached to symplectic representations of various *cyclic* groups over the field $\Bbbk = \mathbb{F}_p$ of p elements. The theory of such representations is developed in [2] and summarized, with some additions and modifications, in §3 of [13]. To prove the desired result, it is necessary to expand that framework a little, to cover symplectic $\Bbbk$-representations of finite *abelian* groups of order prime to p.

11.1. C be a finite abelian group of order not divisible by p. A *symplectic* $\Bbbk C$-*module* is a pair (M,h), consisting of a finite $\Bbbk C$-module M and a nondegenerate, alternating, bilinear form $h : M \times M \to \Bbbk$ which is C-invariant,

$$h(cm_1, cm_2) = h(m_1, m_2), \quad m_i \in M,\ c \in C.$$

Example. Let V be a finite $\Bbbk C$-module. Let $V^* = \mathrm{Hom}(V, \Bbbk)$ and let $\langle\, ,\rangle$ be the canonical pairing $V \times V^* \to \Bbbk$. We impose on V^* the C-action which makes this pairing C-invariant. The module $V \oplus V^*$ carries the bilinear pairing

$$h((v_1, v_1^*), (v_2, v_2^*)) = \langle v_1, v_2^*\rangle - \langle v_2, v_1^*\rangle,$$

which is nondegenerate, alternating and C-invariant. We refer to the pair $H(V) = (V \oplus V^*, h)$ as the *C-hyperbolic space* on V.

Returning to the general situation, the obvious notions of *orthogonal sum* and *C-isometry* apply. A symplectic $\Bbbk C$-module (M,h) is said to be *irreducible* if it cannot be written as an orthogonal sum $(M_1, h_1) \perp (M_2, h_2)$, for non-zero symplectic modules (M_i, h_i). As an instance of the general theory of Hermitian forms over semisimple rings with involution [18], [19], we have the following standard properties.

Lemma.

(1) *If (M,h) is a symplectic $\Bbbk C$-module, then*

$$(M,h) \perp (M,-h) \cong H(M).$$

(2) *Let (M,h) be a symplectic $\Bbbk C$-module. There then exist irreducible symplectic $\Bbbk C$-modules (M_i, h_i), $1 \leqslant i \leqslant r$, such that*

$$(M,h) \cong (M_1, h_1) \perp (M_2, h_2) \perp \cdots \perp (M_r, h_r).$$

The factors (M_i, h_i) *are uniquely determined, up to* C*-isometry and permutation.*

(3) *Let* (M, h) *be an irreducible symplectic* $\Bbbk C$*-module. If* M *is not irreducible as* $\Bbbk C$*-module, there exists an irreducible* $\Bbbk C$*-module* V *such that* $(M, h) \cong H(V)$*. In this case,* $(M, h) \cong H(U)$ *if and only if either* $U \cong V$ *or* $U \cong V^*$*.*

11.2. We consider briefly the structure of irreducible $\Bbbk C$-modules.

Let $\bar{\Bbbk}/\Bbbk$ be an algebraic closure, and set $\Omega = \mathrm{Gal}(\bar{\Bbbk}/\Bbbk)$. Let $\chi : C \to \bar{\Bbbk}^\times$ be a character. In particular, the finite subgroup $\chi(C)$ of $\bar{\Bbbk}^\times$ is *cyclic*. Let $\Bbbk[\chi]$ denote the field extension of $\Bbbk$ generated by the values $\chi(c)$, $c \in C$. The group C acts on $\Bbbk[\chi]$ by $c : x \mapsto \chi(c)x$, and so $\Bbbk[\chi]$ provides a finite $\Bbbk C$-module which we denote V_χ.

Lemma. *The* $\Bbbk C$*-module* V_χ *is irreducible. The map* $\chi \mapsto V_\chi$ *induces a bijection between the set of* Ω*-orbits in* $\mathrm{Hom}(C, \bar{\Bbbk}^\times)$ *and the set of isomorphism classes of irreducible* $\Bbbk C$*-modules.*

Proof. The group algebra $\Bbbk C$ is commutative and semisimple. It is therefore isomorphic to a direct product of finite field extensions of $\Bbbk$. Indeed, $\Bbbk C \cong \prod_\chi \Bbbk[\chi]$, where χ ranges over $\Omega\backslash\mathrm{Hom}(C, \bar{\Bbbk}^\times)$. □

Observe that, in this formulation, $(V_\chi)^* \cong V_{\chi^{-1}}$. We may now list the irreducible symplectic $\Bbbk C$-modules.

Proposition.

(1) *Let* $\chi \in \mathrm{Hom}(C, \bar{\Bbbk}^\times)$*. The symplectic module* $H(V_\chi)$ *is irreducible except when* $\chi \neq \chi^{-1}$ *but* $\chi^{-1} = \chi^\omega$*, for some* $\omega \in \Omega$*.*

(2) *Let* $\chi, \psi \in \mathrm{Hom}(C, \bar{\Bbbk}^\times)$*. The spaces* $H(V_\chi)$*,* $H(V_\psi)$ *are* C*-isometric if and only if* χ *is* Ω*-conjugate to* ψ *or* ψ^{-1}*.*

(3) *Suppose* $\chi \in \mathrm{Hom}(C, \bar{\Bbbk}^\times)$ *satisfies* $\chi^{-1} = \chi^\omega \neq \chi$*, for some* $\omega \in \Omega$*. The module* V_χ *then admits a* C*-invariant, nondegenerate alternating form. This form is uniquely determined up to* C*-isometry.*

Proof. This is effectively the same as the case of C cyclic, treated in [2] (8.2.2), (8.2.3). □

Corollary. *Let (M, h) be an irreducible symplectic $\Bbbk C$-module.*

(1) *If $h' : M \times M \to \Bbbk$ is a C-invariant, nondegenerate, alternating form, then h' is C-isometric to h.*
(2) *The image of C in $\operatorname{Aut} M$ is cyclic.*

Proof. If (M, h) is hyperbolic, assertion (1) is straightforward. Otherwise, $M \cong V_\chi \cong \Bbbk[\chi]$, for some χ. In particular, C acts on V_χ via the cyclic quotient $\chi(C)$ and we are reduced to the case of the proposition.

Assertion (2) is therefore clear in the case where (M, h) is not hyperbolic. If it is hyperbolic, then $M \cong V_\chi \oplus (V_\chi)^*$, for some χ. The group C acts on both factors via the same cyclic quotient $\chi(C) = \chi^{-1}(C)$. $\square$

So, if (M, h) is any symplectic $\Bbbk C$-module, the C-isometry class of h is determined by the $\Bbbk C$-isomorphism class of M. We therefore now tend to suppress explicit mention of the form h.

11.3. Let M be an irreducible symplectic $\Bbbk C$-module. Suppose, for the moment, that C is *cyclic*. Following [13] 3.4 Definition 3, we may define a constant $t^0_C(M) = \pm 1$ and a character $t^1_C(M) : C \to \{\pm 1\}$. We further set $t_C(M) = t^0_C(M)\, t^1_C(M; c_0)$, for any generator c_0 of the cyclic group C. It follows from the definitions in [13] that these quantities depend only on the image of C in Aut M.

We return to the general case of a finite *abelian* group C, of order prime to p. Let M be an irreducible symplectic $\Bbbk C$-module. The image $\overline{C}$ of C in $\operatorname{Aut} M$ is cyclic, by 11.2 Corollary (2). We accordingly define

$$\begin{aligned} t^0_C(M) &= t^0_{\overline{C}}(M), \\ t_C(M) &= t_{\overline{C}}(M), \\ t^1_C(M) &= \text{the inflation to } C \text{ of } t^1_{\overline{C}}(M). \end{aligned}$$

The definitions then yield

$$t_C(M) = t^0_C(M)\, t^1_C(M; c),$$

for any $c \in C$ whose image in $\operatorname{Aut} M$ generates $\overline{C}$.

We extend the definitions in the obvious way: if M is a symplectic $\Bbbk C$-module, we write it as an orthogonal sum $M = M_1 \perp M_2 \perp \cdots \perp M_r$, where the M_i are irreducible symplectic $\Bbbk C$-modules, and set

$$\begin{aligned} t^k_C(M) &= \prod_{i=1}^r t^k_C(M_i), \quad k = 0, 1, \\ t_C(M) &= \prod_{i=1}^r t_C(M_i). \end{aligned} \tag{11.3.1}$$

Proposition. *Let M be an irreducible symplectic $\Bbbk C$-module. If D is a subgroup of C, the space M^D of D-fixed points in M is either $\{0\}$ or M. If $M^D = M$, then*

$$t_D(M) = t^0_C(M)\, t^1_C(M; d),$$

for any $d \in D$ generating the image of D in $\operatorname{Aut} M$.

Proof. Since M^D is a C-subspace of M, the first assertion is clear if M is irreducible as linear $\Bbbk C$-module. Suppose, therefore, that $M = H(V)$, for some irreducible $\Bbbk C$-module V. Thus $M = V \oplus V^*$. Again, V^D is either zero or V. In the first case, $(V^*)^D$ is zero, in the second it is V^*. The first assertion thus follows, and the second is now an instance of 3.5 Proposition 5 of [13]. □

11.4. We return to the situation and notation of 10.4, in order to prove the Symplectic Signs Lemma. In particular, we have chosen a prime element ϖ_F of F, and ϖ is a prime element of E_0 lying in $C_E(\varpi_F)$. We are given an automorphism $\alpha \in \Upsilon = \operatorname{Aut}(E|F)$. The group $C = C_E(\varpi_F)/C_F(\varpi_F)$ is finite abelian, of order relatively prime to p. The space J^1_θ/H^1_θ provides a symplectic $\Bbbk C$-module.

If M is a symplectic $\Bbbk C$-module, we denote by M_α the space of ϖ^α-fixed points in M. In particular, M_1 is the space of ϖ-fixed points. We have to prove:

Proposition. *If M is a symplectic $\Bbbk C$-module, then*

$$t_{\langle\varpi^\alpha\rangle}(M)\, t^0_{\boldsymbol{\mu}_E}(M_\alpha)\, t^1_{\boldsymbol{\mu}_E}(M; \varpi^\alpha/\varpi) = t_{\langle\varpi\rangle}(M)\, t^0_{\boldsymbol{\mu}_E}(M_1). \tag{11.4.1}$$

Proof. The t-invariants are multiplicative relative to orthogonal sums (11.3.1). We henceforward assume, therefore, that the given symplectic $\Bbbk C$-module M is *irreducible.*

Consider first the case in which both ϖ and ϖ^α act trivially on M. Each side of the relation (11.4.1) then reduces to $t^0_{\boldsymbol{\mu}_E}(M)$, and there is nothing to prove.

Suppose next that ϖ acts trivially, while ϖ^α does not. The desired relation is then

$$t_{\langle\varpi^\alpha\rangle}(M)\, t^1_{\boldsymbol{\mu}_E}(M;\varpi^\alpha) = t^0_{\boldsymbol{\mu}_E}(M). \tag{11.4.2}$$

The proposition of 11.3 shows that ϖ^α acts on M with only the trivial fixed point and, further, $t_{\langle\varpi^\alpha\rangle}(M) = t^0_C(M)t^1_C(M;\varpi^\alpha)$. However, since C is generated by $\boldsymbol{\mu}_E$ and ϖ, our hypothesis on ϖ implies $t^j_C(M) = t^j_{\boldsymbol{\mu}_E}(M)$, $j = 0,1$. That is, $t_{\langle\varpi^\alpha\rangle}(M) = t^0_{\boldsymbol{\mu}_E}(M)\, t^1_{\boldsymbol{\mu}_E}(M;\varpi^\alpha)$, whence (11.4.2) follows.

The next case, where ϖ^α acts trivially while ϖ does not, is exactly the same.

We are reduced to the case where neither ϖ nor ϖ^α acts trivially. If $\boldsymbol{\mu}_E$ acts trivially, the result is immediate. We therefore assume it does not. The desired relation is

$$t_{\langle\varpi^\alpha\rangle}(M)\, t^1_{\boldsymbol{\mu}_E}(M;\varpi^\alpha/\varpi) = t_{\langle\varpi\rangle}(M). \tag{11.4.3}$$

If the root of unity ϖ^α/ϖ acts trivially on M, this relation is immediate. We therefore assume it does not. Using 11.3 Proposition as before, we have

$$\begin{aligned} t_{\langle\varpi^\alpha\rangle}(M) &= t^0_C(M)\, t^1_C(M;\varpi^\alpha), \\ t_{\langle\varpi\rangle}(M) &= t^0_C(M)\, t^1_C(M;\varpi). \end{aligned}$$

We therefore need to show

$$t^1_{\boldsymbol{\mu}_E}(M;\varpi^\alpha/\varpi) = t^1_C(M;\varpi^\alpha/\varpi). \tag{11.4.4}$$

If the symplectic $\mathbb{F}_pC$-module M is hyperbolic, (11.4.4) follows directly from the definition of the character $t^1_C(M)$. We therefore assume M is not hyperbolic over C (one says that M is *anisotropic*). Let ζ be some generator of $\boldsymbol{\mu}_E$. By definition, $t^0_C(M) = -1$, so

$$t_{\boldsymbol{\mu}_E}(M) = t^0_C(M)\, t^1_C(M;\zeta) = -t^1_C(M;\zeta).$$

On the other hand,

$$t_{\boldsymbol{\mu}_E}(M) = t^0_{\boldsymbol{\mu}_E}(M)\, t^1_{\boldsymbol{\mu}_E}(M;\zeta).$$

As in 11.2, $M \cong V_\chi = \Bbbk[\chi]$, where $\chi \in \mathrm{Hom}(C, \bar{\Bbbk}^\times)$ satisfies $\chi^{-1} = \chi^\omega \neq \chi$, for some $\omega \in \Omega$. In the present case, the $\Bbbk\boldsymbol{\mu}_E$-module M has length

$$d = \big[\Bbbk[\chi] : \Bbbk[\chi|_{\boldsymbol{\mu}_E}]\big],$$

and is symplectic. Each irreducible linear component is equivalent to $M_0 = \Bbbk[\chi|_{\boldsymbol{\mu}_E}]$. If d is odd, the space M_0 is therefore anisotropic over $\boldsymbol{\mu}_E$. We get $t^0_{\boldsymbol{\mu}_E}(M) = t^0_{\boldsymbol{\mu}_E}(M_0)^d = -1$ and $t^1_{\boldsymbol{\mu}_E}(M) = t^1_C(M)|_{\boldsymbol{\mu}_E}$. The desired relation (11.4.4) follows in this case.

In the final case, where d is even, the symplectic $\Bbbk\boldsymbol{\mu}_E$-module M is the direct sum of $d/2$ copies of the hyperbolic space $H(M_0)$. We assert that $\boldsymbol{\mu}_E$ acts on M_0 via a character of order 2. Let $\Bbbk_0$ denote the subfield of $\Bbbk[\chi]$ for which $[\Bbbk[\chi] : \Bbbk_0] = 2$. The group $\chi(C)$ is then contained in the kernel of the field norm $\mathrm{N}_{\Bbbk[\chi]/\Bbbk_0}$, and this kernel intersects $\Bbbk_0^\times$ in $\{\pm 1\}$. As $\Bbbk_0$ contains $\Bbbk[\chi|_{\boldsymbol{\mu}_E}]$, the restriction of χ to $\boldsymbol{\mu}_E$ has order $\leqslant 2$. Indeed, it has order exactly 2, since we have eliminated the case where $\boldsymbol{\mu}_E$ acts trivially.

In particular, $M_0 \cong \Bbbk$. It follows that $t^1_{\boldsymbol{\mu}_E}(M)$ is non-trivial if and only if $p \equiv 3 \pmod 4$ and $d/2$ is odd (as follows from the definition of t^1 in [13]). On the other hand, this same condition is equivalent to the character $t^1_C(M)$ being non-trivial on $\chi^{-1}(\pm 1)$.

This completes the proof of the proposition. □

We have also completed the proof of the Symplectic Signs Lemma.

12. Main Theorem and examples

We complete the proof of the Comparison Theorem of 7.3, and prove the Types Theorem of 7.6.

12.1. Let $\alpha \in \widehat{\mathcal{P}}_F$ and let $m \geqslant 1$ be an integer. Set $E = Z_F(\alpha)$. Let E_m/E be unramified of degree m and set $\Delta = \mathrm{Gal}(E_m/E)$.

Let $\Sigma \in \mathcal{G}^0_m(F; \mathcal{O}_F(\alpha))$. By 1.3 Proposition, there exists $\rho \in \widehat{\mathcal{W}}_E$ such that $\rho|_{\mathcal{P}_F} \cong \alpha$. Let ρ_m be the restriction of ρ to $\mathcal{W}_{E_m}$. As in 1.6, there exists a Δ-regular character $\xi \in X_1(E_m)$ such that $\Sigma \cong \mathrm{Ind}_{E_m/F}\, \xi \otimes \rho_m$.

Let K_m/F be the maximal unramified sub-extension of E_m/F and put $\sigma = \mathrm{Ind}_{E_m/K_m}\, \xi \otimes \rho_m$. Thus $\sigma \in \mathcal{G}^0_1(K_m; \mathcal{O}_{K_m}(\alpha))$. Since E_m/K_m is totally ramified,

we may apply 8.2 Corollary to get a *unique* character $\mu = \mu_{1,\alpha}^{K_m} \in X_1(E_m)$ such that

$$^L\sigma = \mu \odot_{\Phi_{E_m}(\alpha)} {}^N\sigma, \tag{12.1.1}$$

for all $\sigma \in \mathcal{G}_1^0(K_m; \mathcal{O}_{K_m}(\alpha))$. The field $K = K_m^\Delta/F$ is the maximal unramified sub-extension of E/F. The group Δ acts on both $\mathcal{G}_1^0(K_m; \mathcal{O}_{K_m}(\alpha))$ and $\mathcal{A}_1^0(K_m; \Phi_{K_m}(\alpha))$, via its natural action on K_m. Both the Langlands correspondence and the naïve correspondence preserve these actions (7.1 Proposition, 7.2 Proposition). The uniqueness property of $\mu_{1,\alpha}^{K_m}$ implies it is fixed by Δ, that is,

$$\mu_{1,\alpha}^{K_m} = \mu_{1,\alpha}^{K} \circ \mathrm{N}_{E_m/E}, \tag{12.1.2}$$

for a character $\mu_{1,\alpha}^K \in X_1(E)$, uniquely determined modulo $X_0(E)_m$.

We remark on one consequence of (12.1.1). Comparing central characters, as in 7.2, we get:

Proposition. *The character $\mu_{1,\alpha}^{K_m}$ agrees on U_{K_m} with the discriminant character d_{E_m/K_m} of the extension E_m/K_m. The character $\mu_{1,\alpha}^K$ agrees on U_K with $d_{E/K}$.*

12.2. For the next step, we have $\Sigma = \mathrm{Ind}_{K_m/F}\,\sigma$, so

$$^L\Sigma = \mathrm{A}_{K_m/F}\, {}^L\sigma, \quad {}^N\Sigma = ind_{K_m/F}\, {}^N\sigma.$$

In particular,

$$^L\Sigma = \mu_{1,\alpha}^K \odot_{\Phi_E(\alpha)} \mathrm{A}_{K_m/F}\, {}^N\sigma, \tag{12.2.1}$$

by (9.1.3). The Unramified Induction Theorem of 9.1 (or 9.2 Theorem) gives a unique character $\mu_\alpha^{K_m/F} \in X_1(E_m)^\Delta$ such that

$$\mathrm{A}_{K_m/F}\,\tau = ind_{K_m/F}\,\mu_\alpha^{K_m/F} \odot_{\Phi_{E_m}(\alpha)} \tau,$$

for every Δ-regular $\tau \in \mathcal{A}_1^0(K_m; \Phi_{K_m}(\alpha))$. We write

$$\mu_\alpha^{K_m/F} = \mu_{m,\alpha}^{K/F} \circ \mathrm{N}_{E_m/E},$$

for a character $\mu^{K/F}_{m,\alpha} \in X_1(E)$, uniquely determined modulo $X_0(E)_m$. Applying (9.1.3), we find

$$ {}^{L}\Sigma = \mu^{K/F}_{m,\alpha}\, \mu^{K}_{1,\alpha} \odot_{\Phi_E(\alpha)} {}^{N}\Sigma, \tag{12.2.2} $$

and this relation holds for all $\Sigma \in \mathcal{G}^0_m(F;\Phi_F(\alpha))$. The character

$$ \mu^{F}_{m,\alpha} = \mu^{K/F}_{m,\alpha}\, \mu^{K}_{1,\alpha} \in X_1(E) \tag{12.2.3} $$

is uniquely determined modulo $X_0(E)_m$, and satisfies the requirements of the Comparison Theorem. □

12.3. We prove the Types Theorem of 7.6. We use the notation of the statement. In addition, we let E_m/F be the maximal tamely ramified sub-extension of P_m/F and $K = E \cap K_m$

The representation ${}^{N}\Sigma$ contains an extended maximal simple type $\Lambda_0 \in \mathcal{T}(\theta)$ of the form $\lambda_\tau \ltimes \nu_0$, where $\nu_0 \in \mathcal{H}(\theta)$ satisfies $\det \nu_0(\zeta) = 1$, for all $\zeta \in \boldsymbol{\mu}_{K_m}$. By the Comparison Theorem, the representation ${}^{L}\Sigma$ contains the representation $\mu^{F}_{m,\alpha} \odot \Lambda_0 \in \mathcal{T}(\theta)$.

We evaluate $\mu^{F}_{m,\alpha}$ on units by first using the decomposition (12.2.3). The factor $\mu^{K}_{1,\alpha}$ is $d_{E/K}$, which is unramified if $[E{:}K]$ is odd, of order 2 on U_K otherwise. The other factor of $\mu^{F}_{m,\alpha}$ is given by (10.7) Corollary, and the result then follows from 5.4 Lemma. □

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